STRATIFICATION OF FREE BOUNDARY POINTS FOR A TWO-PHASE VARIATIONAL PROBLEM

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Abstract. In this paper we study the two-phase Bernoulli type free boundary problem arising from the minimization of the functional

$$J(u) := \int_\Omega |\nabla u|^p + \lambda_+^p \chi_{\{u>0\}} + \lambda_-^p \chi_{\{u\le 0\}}, \quad 1 < p < \infty.$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $\lambda_\pm$ are positive constants such that $\lambda_+^p - \lambda_-^p > 0$. We prove the following dichotomy: if x_0 is a free boundary point then either the free boundary is smooth near x_0 or u has linear growth at x_0. Furthermore, we show that for $p > 1$ the free boundary has locally finite perimeter and the set of non-smooth points of free boundary is of zero $(N-1)$-dimensional Hausdorff measure. Our approach is new even for the classical case $p = 2$.

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1. Introduction

In this paper we study the local minimizers of

$$J(u) := \int_\Omega |\nabla u|^p + \lambda_+^p \chi_{\{u>0\}} + \lambda_-^p \chi_{\{u\le 0\}}, \quad u \in \mathcal{A}, \tag{1.1}$$

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where Ω is a bounded and smooth domain in $\mathbb{R}^N$, χ_D is the characteristic function of the set $D \subset \mathbb{R}^N$, and $\lambda_\pm$ are positive constants such that

$$\Lambda := \lambda_+^p - \lambda_-^p > 0. \tag{1.2}$$

The class of admissible functions $\mathcal{A}$ consists of those functions $u \in W^{1,p}(\Omega)$, with $1 < p < \infty$, such that $u - g \in W_0^{1,p}(\Omega)$ for a given boundary datum g.

This type of problems arises in jet flow models with two ideal fluids, see e.g. [4] and [20] page 126, and has been studied in [1] for $p = 2$. When the velocity $\mathbf{v}$ of the planar flow depends on the gradient of the stream function u in power law $\mathbf{v} = |\nabla u|^{p-2}\nabla u$ (see [3]), then the resulted problem for steady state admits a variational formulation with the functional (1.1). In higher dimensions, this models heat (or electrostatic) energy optimization under power Fourier law, see [26].

For admissible functions in $\mathcal{A}^+ = \{u \in \mathcal{A}, u \geq 0\}$ the analogous problem has been studied in [11]. However, the two-phase problem for general growth functionals has remained fundamentally open. Towards this direction there are only some partial results available under the assumption of small Lebesgue density on the negative phase, see [22, 5]. This is due to the lack of a monotonicity formula for $p \neq 2$. However, some weak form of monotonicity type formula is known for the modified Alt-Caffarelli-Friedman functional, namely a discrete monotonicity formula in two spatial dimensions when p is close to 2, see [15].

The aim of this paper is twofold and contributes into the regularity theory of the two-phase free boundary problems: first, we define a suitable notion of flatness for free boundary points which allows to partition the set $\partial\{u > 0\}$ into to disjoint subsets $\mathcal{F}$ and $\mathcal{N}$. Here $\mathcal{F}$ is the set of flat free boundary points and $\mathcal{N}$ the set of non-flat points. These sets are determined by the critical flatness constant h_0, such that if the flatness at $x \in \partial\{u > 0\}$ is less that h_0 then the free boundary must be regular in some vicinity of x. Consequently we can stratify the free boundary points and prove linear growth at the non-flat points of free boundary (see Section 2 for precise definitions and statements).

The advantage of this approach is that it avoids using the optimal regularity for u everywhere and hence circumvents the obstacle imposed by the lack of monotonicity formula. However, our technique renders the local Lipschitz continuity using a simple consequence of Theorem A below. Observe that the non-flat points $x \in \mathcal{N}$ are more interesting to study and it is vital to have linear growth at such points x in order to classify the blow-up profiles.

Second, to study the flat points $x \in \mathcal{F}$ we apply the regularity theory developed for viscosity solutions of two-phase free boundary problems. To do so we prove that any local minimizer is also a viscosity solution. At flat points we get that the free boundary $\partial\{u > 0\}$ is very close to a plane in a suitable coordinate system. Consequently, u must be $\varepsilon-$monotone with $\varepsilon > 0$ small, which in turn implies that the free boundary is $C^{1,\alpha}$ in some vicinity of x. This approach, which is based on the fusion of variational and viscosity solutions, appears to be new and very useful.

Finally, from here we conclude the partial regularity of $\partial\{u > 0\}$, that is $\partial\{u > 0\}$ is countably rectifiable and $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure.

It is worthwhile to point out that our approach is new even for the classical case $p = 2$.

In the forthcoming Section 2 we give the precise statements of the results that we prove. A detailed plan on the organization of the paper will be presented at the end of Section 2.

Basic Notations

$C, C_0, C_n, \cdots$	generic constants,
$\overline{U}$	the closure of a set U,
∂U	the boundary of a set U,
$B_r(x), B_r$	the ball centered at x with radius $r > 0$, $B_r := B_r(0)$,
$\Gamma = \partial\{u > 0\}$	the free boundary $\partial\{u > 0\}$,
$⨍$	mean value integral,
ω_N	the volume of unit ball,
$\Omega^+(u) := \{u > 0\}$	the positivity set of u,
$\Omega^-(u) := \{u < 0\}$	the negativity set of u,
$\mathcal{N}$	the set of non-flat free boundary points, see Definition 2.1,
$\mathcal{F}$	$\partial\{u > 0\} \setminus \mathcal{N}$,
$\lambda(u)$	$\lambda_+^p \chi_{\{u>0\}} + \lambda_-^p \chi_{\{u\le 0\}}$,
Λ, Λ_0	$\Lambda = \lambda_+^p - \lambda_-^p$, $\Lambda_0 = \frac{\Lambda}{p-1}$ the Bernoulli constants.

2. Main Results

2.1. **Setup.** The existence of bounded minimizers of the functional in (1.1) can be easily established using the semicontinuity of the $p-$Dirichlet energy and the weak convergence in $W^{1,p}$, and can be found in [11].

Let now $x_0 \in \partial\{u > 0\}$ and

$$\mathcal{S}(h; x_0, \nu) := \{x \in \mathbb{R}^n : -h < (x - x_0) \cdot \nu < h\} \tag{2.1}$$

be the slab of height $2h$ in unit direction ν. Let $h_{\min}(x_0, r, \nu)$ be the minimal height of the slab containing the free boundary in $B_r(x_0)$, i.e.

$$h_{\min}(x_0, r, \nu) := \inf\{h : \partial\{u > 0\} \cap B_r(x_0) \subset S(h; x_0, \nu) \cap B_r(x_0)\}. \tag{2.2}$$

Put

$$h(x_0, r) := \inf_{\nu \in \mathbb{S}^N} h_{\min}(x_0, r, \nu). \tag{2.3}$$

Clearly $h(x_0, r)$ is **non-decreasing** in r.

Theorem A. *Let u be a local minimizer of* (1.1). *Then, for any bounded subdomain $D \Subset \Omega$ there are positive constants h_0 and L depending only on $N, p, \Lambda, \sup_\Omega |u|$ and $\operatorname{dist}(\partial\Omega, D)$ such that, for any $x_0 \in D \cap \partial\{u > 0\}$ one of the following two alternatives holds:*

- *if $h(x_0, 2^{-k}) \ge h_0 2^{-k-1}$, for all $k \in \mathbb{N}, 2^{-k} < \operatorname{dist}(\partial\Omega, D)$, then*

$$\sup_{B_{r/2}(x_0)} |u| \le Lr,$$

for all $0 < r < \operatorname{dist}(\partial\Omega, D)$,

- *if* $h(x_0, 2^{-k_0}) < h_0 2^{-k_0-1}$ *for some* $k_0 \in \mathbb{N}$ *then the free boundary* $\partial\{u > 0\}$ *is* $C^{1,\alpha}$ *in some neighbourhood of* x_0.

We call $h_0/2$ **the critical flatness constant**.

The statement in Theorem A leads to the following definition:

Definition 2.1. *We say that* $z \in \partial\{u > 0\}$ *is non-flat if* $h(z, 2^{-k}) \geq h_0 2^{-k-1}$ *for all* $k \in \mathbb{N}$ *such that* $2^{-k} < \operatorname{dist}(z, \partial\Omega)$. *The set of all non-flat points is denoted by* $\mathcal{N}(\Gamma)$ *or* $\mathcal{N}$ *for short.*

Notice that if $z \notin \mathcal{N}$ then $h(z, 2^{-k_0}) < h_0 2^{-k_0-1}$, for some $k_0 \in \mathbb{N}$. So Theorem A gives a partition of the free boundary of the form

$$\partial\{u > 0\} = \mathcal{F} \cup \mathcal{N} \tag{2.4}$$

where $\mathcal{F} := \left\{x \in \partial\{u > 0\} : h(x, 2^{-k_0}) < h_0 2^{-k_0-1}, \text{ for some } k_0 \in \mathbb{N}\right\}$ is the set of flat free boundary points.

Theorem B. *Let* u *be as in Theorem A. Then, for any subdomain* $D \Subset \Omega$ *we have*

$$\mathcal{H}^{N-1}(\partial\{u > 0\} \cap D) < \infty$$

and

$$\mathcal{H}^{N-1}\big((\partial\{u > 0\} \setminus \partial_{red}\{u > 0\}) \cap D\big) = 0.$$

In particular, $\mathcal{H}^{N-1}(D \cap \mathcal{N}) = 0$.

We remark that, as a consequence of Theorem A, we also obtain local Lipschitz continuity for the minimizers.

Theorem C. *Let* u *be as in Theorem A. Then for any subdomain* $D \Subset \Omega$ *there is a constant* C_0 *depending only on* $\operatorname{dist}(D, \partial\Omega), N, p, \Lambda, h_0$ *and* L *such that*

$$|u(x) - u(y)| \leq C_0 |x - y|, \quad \forall x, y \in D. \tag{2.5}$$

The proof of Theorem C will be given in Section 9.

2.2. **Strategy of the proofs.** The methods and the techniques that we employ to prove Theorems A and B pave the way to a number of new approaches.

First, we fuse the variational methods with the viscosity theory. This is done by proving that any local minimizer $u \in W^{1,p}(\Omega)$ is also a viscosity solution (see Section 4, and in particular Theorem 4.2). The key ingredient in the proof is the linear development of a nonnegative $p-$harmonic function v in $D \subset \mathbb{R}^N$ near $x_0 \in \partial D$ that vanishes continuously on $B_r(x_0) \cap \partial D$, see Lemma 4.3. There is a subtle point in the proof of the linear development lemma which amounts to the following claim: if $x_0 \in \partial\{u > 0\}$ and $B_r(y_0) \subset \{u > 0\}$ with $x_0 \in \partial B_r(y_0)$ then u has linear growth near x_0, i.e. there is a constant $C(x_0) > 0$ (depending on x_0) such that $|u(x)| \leq C(x_0)|x - x_0|$ near x_0. Indeed, by standard barrier argument we have that

$$u^-(x) \leq \sup_{B_{2r}(y_0)} u^- \frac{\Phi(|x - y_0|) - \Phi(r)}{\Phi(2r) - \Phi(r)}$$

where $\Phi(t) = t^{\frac{p-N}{p-1}}$. Therefore u^- has linear growth near x_0. Now the linear growth of u^+ near x_0 follows form Lemma 3.7. Clearly the same claim is valid if $B_r(y_0) \subset \{u < 0\}$ and

$x_0 \in \partial B_r(y_0)$. We stress on the fact that Lemma 4.3 on linear development remains valid for solutions to a wider class of equations for which Harnack's inequality and Hopf's Lemma are valid.

Second, we compare $r = 2^{-k}$ with the minimal height $h(x_0, r)$ of the parallel slab of planes containing $B_r(x_0) \cap \partial\{u > 0\}$, for $x_0 \in \partial\{u > 0\}$. More precisely, take $k \in \mathbb{N}$, and fix $h_0 > 0$, then

$$\text{either } h(x_0, 2^{-k}) \geq h_0 2^{-k-1}, \tag{2.6}$$

or

$$h(x_0, 2^{-k}) < h_0 2^{-k-1}. \tag{2.7}$$

Consequently, for given $x_0 \in \partial\{u > 0\}$ there are two alternatives: either for some k we arrive at (2.7) and this will mean that x_0 is a flat point of $\partial\{u > 0\}$ (if $h_0 > 0$ is small) or (2.6) holds for sufficiently large $k \geq k_0$. The latter implies linear growth at x_0. Note that the non-flat points are more interesting to study and having the linear growth at such points allows one to use compactness argument and blow-up u in order to study the properties of the resulted configuration as done in the proofs of (7.2), (7.3) and (7.6). Note that if (2.6) holds for $1 \leq k < k_0$ then we have linear growth for u near x_0 unto the level 2^{-k_0}, see Corollary 6.2.

Altogether, this approach allows us to prove the main properties of the free boundary without using the full optimal regularity of u and can be applied to a wide class of variational free boundary problems with two phases. A diagram showing the scheme of the proof is given below.

As for the proof of the partial regularity result, i.e.

$$\mathcal{H}^{N-1}\left(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}\right) = 0,$$

we employ a non-degeneracy result obtained in Proposition 3.5 for u^+ and some estimates for the Radon measure $\Delta_p u^+$ given in Lemma 7.1. This is a standard approach but more involved because the linear growth is valid only at non-flat points of the free boundary.

2.3. **Structure of the paper.** In Section 3 we collect some material, mostly of technical nature, that we will use in the other sections. In particular, we prove the continuity of minimizers, by showing that $\nabla u \in BMO_{loc}$ if $p > 2$ and $|\nabla u|^{\frac{p}{2}-1}\nabla u \in BMO_{loc}$ if $1 < p < 2$. We also recall the Liouville's Theorem and some basic properties of minimizers. Finally we show that u^+ is non-degenerate, in the sense of Proposition 3.5, and a coherence lemma (see Lemma 3.7).

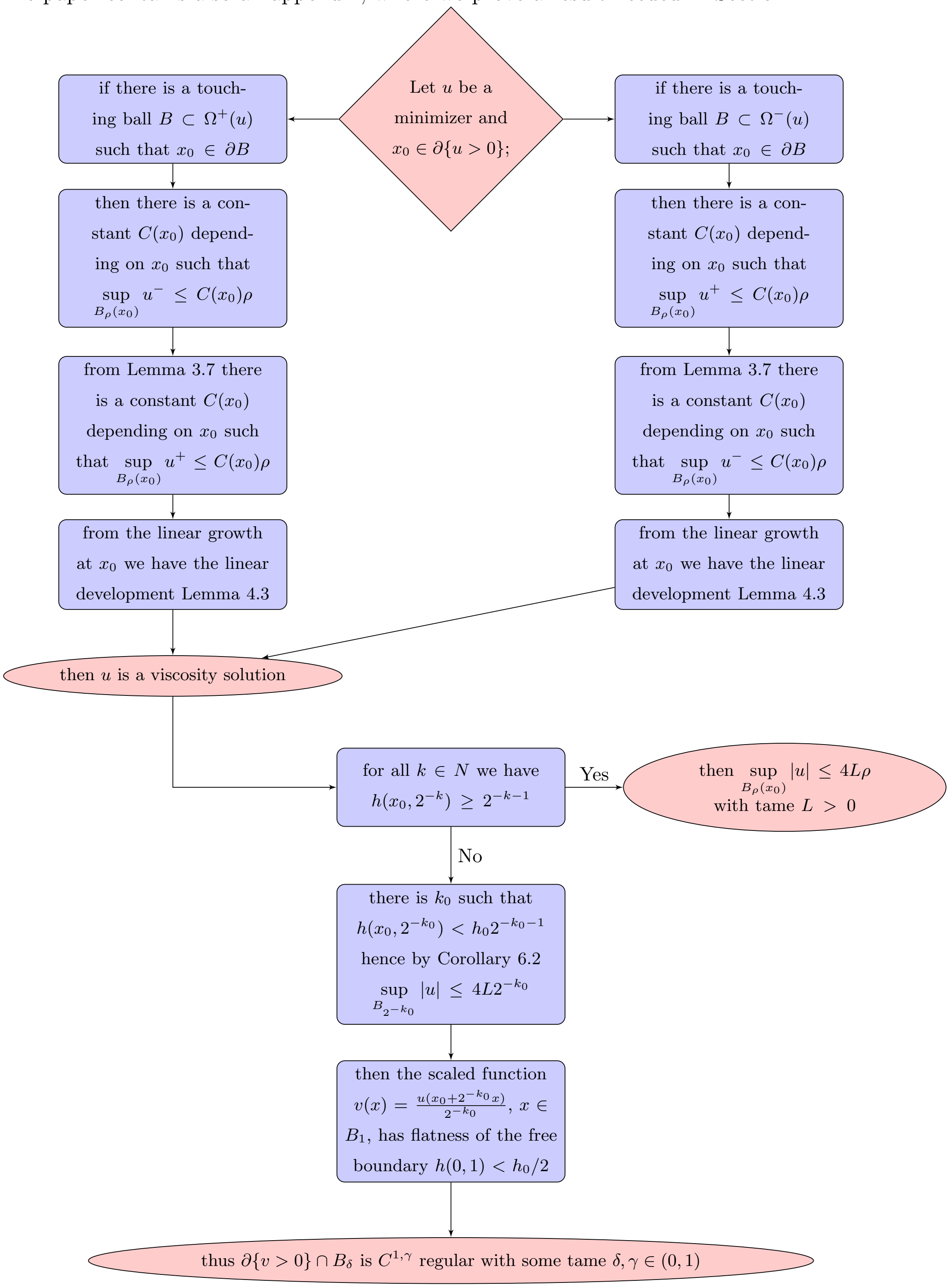
In Section 4 we prove that any minimizer of the functional in (1.1) is also a viscosity solution, according to Definition 4.1. This will allow us to apply the regularity theory developed in [23, 24] for viscosity solutions and infer that the free boundary is $C^{1,\alpha}$ regular near flat points.

In Section 5 we discuss and compare the notions of ε-monotonicity of minimizers and of slab flatness of the free boundary.

Section 6 is devoted to the proof of Theorem A and Section 7 contains the set up for the proof of Theorem B. In Section 8 we deal with the blow-up of minimizers proving some useful convergence and finish the proof of Theorem B.

Then in Section 9 we prove Theorem C.

The paper contains also an appendix, where we prove a result needed in Section 4.



3. Technicalities

In this section we prove some basic properties of minimizers.

3.1. **A BMO estimate for ∇u.** We first prove the continuity of minimizers of (1.1) with any $\alpha-$Hölder modulus of continuity, with $\alpha \in (0, 1)$, if $p \in (1, 2)$ and log-Lipschitz modulus of

continuity if $p>2$. Our method is a variation of [1] and uses some standard inequalities for the functionals with $p-$power growth.

Lemma 3.1 (Continuity of minimizers)**.** *Let u be a minimizer of* (1.1)*. Then*

- *for $1<p<2$, we have that $|\nabla u|^{\frac{p-2}{2}}\nabla u\in BMO(D)$ for any bounded subdomain $D\Subset\Omega$, and consequently $u\in C^\sigma(D)$ for any $\sigma\in(0,1)$,*
- *for $p>2$, we have that $\nabla u\in BMO(D)$, for any bounded subdomain $D\Subset\Omega$, and thus u is locally log-Lipschitz continuous.*

In particular, $\nabla u\in L^q(D)$ for any $1<q<\infty$ and for any $p>1$.

Proof. Fix $R\geq r>0$ and $x_0\in D$ such that $B_{2R(x_0)}\Subset D$. Let v be the solution of

$$\begin{cases}\Delta_p v=0 & \text{in } B_{2R}(x_0),\\ v=u & \text{on } \partial B_{2R}(x_0).\end{cases}$$

Comparing $J(u)$ with $J(v)$ in $B_{2R}(x_0)$ yields

$$\begin{aligned}\int_{B_{2R}(x_0)}|\nabla u|^p-|\nabla v|^p &\leq\int_{B_{2R}(x_0)}\lambda_+^p\chi_{\{v>0\}}+\lambda_-^p\chi_{\{v\leq 0\}}-\left(\lambda_+^p\chi_{\{u>0\}}+\lambda_-^p\chi_{\{u\leq 0\}}\right)\\ &\leq CR^N,\end{aligned}\tag{3.1}$$

for some $C>0$. On the other hand, the following estimate is true (see [11] page 100)

$$\int_{B_{2R}(x_0)}|\nabla u|^p-|\nabla v|^p\geq\gamma\begin{cases}\int_{B_{2R}(x_0)}(|\nabla u|+|\nabla v|)^{p-2}|\nabla(u-v)|^2, & \text{if } 1<p<2,\\ \int_{B_{2R}(x_0)}|\nabla(u-v)|^p, & \text{if } p>2,\end{cases}\tag{3.2}$$

for some tame constant $\gamma>0$ depending on N and p.

Introduce the function $V:\mathbb{R}^N\to\mathbb{R}^N$ defined as follows

$$V(\xi):=\begin{cases}|\xi|^{\frac{p-2}{2}}\xi, & \text{if } 1<p<2,\\ \xi, & \text{if } p>2,\end{cases}\tag{3.3}$$

then from the basic inequalities

$$c^{-1}(|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2\leq|V(\xi)-V(\eta)|^2\leq c(|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2,\tag{3.4}$$

that are valid for any $p>1$ (see [16] page 240), we infer the estimate

$$\int_{B_{2R}(x_0)}|V(\nabla u)-V(\nabla v)|^2\leq CR^N,\tag{3.5}$$

up to renaming C. Indeed, the case $1<p<2$ follows from the second inequality in (3.4). As for the remaining case $p>2$ we have by Hölder's inequality

$$\left(\fint_{B_{2R}(x_0)}|\nabla u-\nabla v|^p\right)^{\frac{1}{p}}\geq\left(\fint_{B_{2R}(x_0)}|\nabla u-\nabla v|^2\right)^{\frac{1}{2}}$$

and (3.5) follows.

Furthermore, for any $\rho>0$, we set

$$(V(\nabla u))_{x_0,\rho}:=\fint_{B_\rho(x_0)}V(\nabla u).$$

Then, from Hölder's inequality we have

$$
\begin{aligned}
|(V(\nabla v))_{x_0,r} - (V(\nabla u))_{x_0,r}|^2 &\le \left(\fint_{B_r(x_0)} |V(\nabla v) - V(\nabla u)| \right)^2 \\
&\le \fint_{B_r(x_0)} |V(\nabla v) - V(\nabla u)|^2.
\end{aligned} \tag{3.6}
$$

We would also need the following estimate for a $p-$harmonic function v: there is $\alpha > 0$ such that for all balls $B_{2R(x_0)} \Subset D$, with $R \ge r > 0$, there exists a universal constant $c > 0$ such that the following Campanato type estimate is valid

$$
\fint_{B_r(x_0)} |V(\nabla v) - V((\nabla v)_{x_0,r})|^2 \le c \left(\frac{r}{R}\right)^{\alpha} \fint_{B_R(x_0)} |V(\nabla v) - V((\nabla v)_{x_0,R})|^2. \tag{3.7}
$$

See [14] Theorem 6.4 for $V(\nabla v) = |\nabla v|^{\frac{p-2}{2}} \nabla v$ and [13] Theorem 5.1 for $V(\nabla v) = \nabla v$.

Denote $\|\cdot\|_{L^2(B_r(x_0))} = \|\cdot\|_{2,r}$, then, using (3.6), we obtain

$$
\begin{aligned}
\|V(\nabla u) - (V(\nabla u))_{x_0,r}\|_{2,r} &\le \|V(\nabla u) - V(\nabla v)\|_{2,r} + \|V(\nabla v) - (V(\nabla v))_{x_0,r}\|_{2,r} \\
&\quad + \|(V(\nabla v))_{x_0,r} - (V(\nabla u))_{x_0,r}\|_{2,r} \\
&\le 2\,\|V(\nabla u) - V(\nabla v)\|_{2,r} + \|V(\nabla v) - (V(\nabla v))_{x_0,r}\|_{2,r} \\
&\le 2\,\|V(\nabla u) - V(\nabla v)\|_{2,r} + C\left(\frac{r}{R}\right)^{\frac{N+\alpha}{2}} \|V(\nabla v) - (V(\nabla v))_{x_0,R}\|_{2,R}, \qquad (3.8)
\end{aligned}
$$

where, in order to get (3.8), we used Campanato type estimate (3.7).

From the triangle inequality for L^2 norm we have

$$
\|V(\nabla v) - (V(\nabla v))_{x_0,R}\|_{2,R} \le 2\,\|V(\nabla u) - V(\nabla v)\|_{2,R} + \|V(\nabla u) - (V(\nabla u))_{x_0,R}\|_{2,R},
$$

and so, combining this with (3.5), we obtain

$$
\begin{aligned}
\|V(\nabla u) - (V(\nabla u))_{x_0,r}\|_{2,r} &\le 2\,\|V(\nabla u) - V(\nabla v)\|_{2,r} \\
&\quad + C\left(\frac{r}{R}\right)^{\frac{N+\alpha}{2}} \left[2\|V(\nabla u) - V(\nabla v)\|_{2,R} + \|V(\nabla u) - (V(\nabla u))_{x_0,R}\|_{2,R}\right] \\
&\le C\left\{\|V(\nabla u) - V(\nabla v)\|_{2,R} + \left(\frac{r}{R}\right)^{\frac{N+\alpha}{2}} \|V(\nabla u) - (V(\nabla u))_{x_0,R}\|_{2,R}\right\} \\
&\le A\left(\frac{r}{R}\right)^{\frac{N+\alpha}{2}} \|V(\nabla u) - V((\nabla u))_{x_0,R}\|_{2,R} + BR^{\frac{N}{2}},
\end{aligned}
$$

for some tame positive constants A and B.

Introduce

$$
\varphi(r) := \sup_{t \le r} \|V(\nabla u) - (V(\nabla u))_{x_0,t}\|_{2,t},
$$

then the former inequality can be rewritten as

$$
\varphi(r) \le A\left(\frac{r}{R}\right)^{\frac{N+\alpha}{2}} \varphi(R) + BR^{\frac{N}{2}},
$$

with some positive constants A, B, α. Applying Lemma 2.1 from [18] Chapter 3, we conclude that there exist $R_0 > 0$ and $c > 0$ such that

$$\varphi(r) \leq c r^{\frac{N}{2}} \left(\frac{\varphi(R)}{R^{\frac{N}{2}}} + B \right),$$

for all $r \leq R \leq R_0$, and hence

$$\int_{B_r(x_0)} |V(\nabla u) - (V(\nabla u))_{x_0,r}|^2 \leq C r^N,$$

for some tame constant $C > 0$. This shows that $V(\nabla u)$ is locally BMO. The log-Lipschitz estimate for $p > 2$ now follows from [10] Theorem 3. The Hölder continuity follows from Sobolev's embedding and the John-Nirenberg Lemma. $\square$

Remark 3.2. *From Lemma 3.1 it follows that for any $D \Subset \Omega$ there is a constant $C > 0$ depending only on $N, p, \Lambda, \sup_\Omega |u|$ and $\operatorname{dist}(D, \partial\Omega)$ such that if $p > 2$ and $x_0 \in \Gamma$, then*

$$\left| \fint_{\partial B_r(x_0)} u \right| \leq C r \quad \textit{for any } B_r(x_0) \subset D,$$

see [15].

3.2. **Liouville's Theorem.** This section is devoted to Liouville's Theorem, that we use in the proof of Proposition 6.1. We add the proof here.

Theorem 3.3. *Let U be a p-harmonic function in $\mathbb{R}^N$ such that*

$$|U(x)| \leq C|x|, \quad \textit{for any } x \in \mathbb{R}^N, \tag{3.9}$$

for some $C > 0$. Then U is a linear function in $\mathbb{R}^N$.

Proof. For any $r > 0$, we introduce the scaled function

$$U_r(x) := \frac{U(rx)}{r}. \tag{3.10}$$

Hence U_r is a p-harmonic function and

$$|U_r(x)| \leq C|x|, \quad \text{for any } x \in \mathbb{R}^N,$$

thanks to (3.9).

Moreover, from the $C^{1,\alpha}$-estimates for p-harmonic functions in B_1 (for some $\alpha \in (0,1)$), see [27], we have that $\sup_{B_1} |\nabla U_r(x)| \leq M$ and, moreover,

$$\frac{|\nabla U_r(x) - \nabla U_r(y)|}{|x-y|^\alpha} = \frac{|\nabla U(rx) - \nabla U(ry)|}{|x-y|^\alpha} \leq M, \quad x, y \in B_1, x \neq y \tag{3.11}$$

for a positive constant M, depending only on N, p and $\sup_{B_2} |U_r(x)| \leq 2C$.

Hence, taking $\xi := rx$ and $\eta := ry$ in (3.11), we obtain that for any $r > 0$

$$|\nabla U(\xi) - \nabla U(\eta)| \leq \frac{M}{r^\alpha} |\xi - \eta|^\alpha, \quad \text{for any } \xi, \eta \in B_r. \tag{3.12}$$

In particular, (3.12) holds true for any $r > 1$. Therefore, letting $\xi, \eta \in B_1$ and sending $r \to +\infty$ in the formula above, we obtain that

$$|\nabla U(\xi) - \nabla U(\eta)| = 0 \quad \text{for any } \xi, \eta \in B_1.$$

Hence, U is linear in B_1. This completes the proof in view of the Unique Continuation Theorem [19]. □

3.3. **Some basic properties of the local minimizers of J.**

Proposition 3.4. *Let $u \in W^{1,p}$ be a local minimizer of* (1.1). *Then*

P.1 $\Delta_p u^{\pm} \geq 0$ *in the sense of distributions and* $\Delta_p u = 0$ *in* $\{u > 0\} \cup \{u < 0\}$,

P.2 *for any* $D \Subset \Omega$ *there is* $c_0 > 0$ *depending only on* $N, p, \Lambda, \sup_\Omega |u|$ *and* $\operatorname{dist}(D, \partial\Omega)$ *such that if*

$$\limsup_{r \to 0} \frac{|B_r(x_0) \cap \{u < 0\}|}{|B_r(x_0)|} \leq c_0, \quad x_0 \in \Gamma \cap D$$

then $\sup_{B_r(x_0)} |u| \leq \frac{C}{c_0} r$ *where* C *is a tame constant.*

Proof. P.1 follows from a standard comparison of u and $u + \varepsilon\varphi$, where φ is a suitable smooth and compactly supported function. P.2 follows from [22]. □

3.4. **A remark on the volume term and scaling.**

It is convenient to define

$$\lambda(u) := \lambda_+^p \chi_{\{u>0\}} + \lambda_-^p \chi_{\{u \leq 0\}} = \Lambda \chi_{\{u>0\}} + \lambda_-^p, \tag{3.13}$$

with $\Lambda := \lambda_+^p - \lambda_-^p > 0$. As a consequence, the functional in (1.1) can be rewritten in an equivalent form

$$J(u) = \int_\Omega |\nabla u|^p + \Lambda \chi_{\{u>0\}} + \lambda_-^p |\Omega|. \tag{3.14}$$

Notice that the last term does not affect the minimization problem, and so if u is a minimizer for J, then it is also a minimizer for

$$\tilde{J}(u) := \int_\Omega |\nabla u|^p + \Lambda \chi_{\{u>0\}}. \tag{3.15}$$

Observe that if $\Lambda > 0$ then the free boundary $\partial\{u > 0\} \cup \partial\{u < 0\}$ for the minimizer u of J coincides with $\partial\{u > 0\}$. Indeed, let $\Gamma_0 := \partial\{u < 0\} \setminus \partial\{u > 0\}$, then we clearly have that if $x_0 \in \Gamma_0$ then there is $r > 0$ such that $u \leq 0$ in $B_r(x_0)$, and so u is $p-$superharmonic in $B_r(x_0)$. On the other hand, we have that $\Lambda = \lambda_+^p - \lambda_-^p > 0$, and so we get a contradiction with P.1 of Proposition 3.4. Therefore $\Gamma_0 = \emptyset$.

The functional $\tilde{J}$ preserves the minimizers under certain scaling. This property is a key ingredient in a number of arguments to follow.

More precisely, let u be a minimizer of (1.1), and take $x_0 \in \partial\{u > 0\}$ and $r > 0$ such that $B_r(x_0) \subset \Omega$. Fixed $\rho > 0$, set also $u_\rho(x) := \frac{u(x_0+\rho x)}{S}$, for some constant $S > 0$. Then one can readily verify that

$$\int_{B_1} |\nabla u_\rho(x)|^p + \left[\frac{\rho}{S}\right]^p \Lambda \chi_{\{u_\rho>0\}} = \left[\frac{\rho}{S}\right]^p \frac{1}{\rho^N} \int_{B_\rho(x_0)} |\nabla u|^p + \Lambda \chi_{\{u>0\}}. \tag{3.16}$$

In particular if we let $S = \rho$ then

$$\int_{B_1} |\nabla u_\rho(x)|^p + \Lambda \chi_{\{u_\rho>0\}} = \frac{1}{\rho^N} \int_{B_\rho(x_0)} |\nabla u|^p + \Lambda \chi_{\{u>0\}}. \tag{3.17}$$

Therefore if u is minimizer of $\tilde{J}$ in $B_\rho(x_0)$ then the scaled function u_ρ is a minimizer of $\tilde{J}$ in B_1.

3.5. **Strong Non-degeneracy.** In this section we deal with a strong form of non-degeneracy for minimizers of (1.1). For $p = 2$, this result is contained in [1] (see in particular Theorem 3.1 there). We use a modification of an argument from [2] Lemma 2.5.

Proposition 3.5. *For any $\kappa \in (0,1)$ there exists a constant $c_\kappa > 0$ such that for any local minimizer of* (1.1) *and for any small ball $B_r \subset \Omega$*

$$\text{if} \quad \frac{1}{r}\left(\fint_{B_r}(u^+)^p\right)^{\frac{1}{p}} < c_\kappa \ \text{then } u \equiv 0 \ \text{in } B_{\kappa r}. \tag{3.18}$$

Proof. By scale invariance of the problem we take $r = 1$ for simplicity and put

$$\varepsilon := \frac{1}{\sqrt{\kappa}} \sup_{B_{\sqrt{\kappa}}} u^+. \tag{3.19}$$

Since u^+ is $p-$subharmonic (recall P.1 in Proposition 3.4), then by [25] Theorem 3.9

$$\varepsilon \le \frac{1}{\sqrt{\kappa}} \frac{C(p,N)}{(1-\sqrt{\kappa})^{\frac{N}{p}}} \left(\fint_{B_1}(u^+)^p\right)^{\frac{1}{p}}.$$

Introduce

$$v(x) := \begin{cases} C_1\varepsilon\left[e^{-\mu|x|^2} - e^{-\mu\kappa^2}\right] & \text{in } B_{\sqrt{\kappa}} \setminus B_\kappa, \\ 0 & \text{in } B_\kappa, \end{cases}$$

where $\mu > 0$ and C_1 is chosen so that

$$v|_{\partial B_{\sqrt{\kappa}}} := \sqrt{\kappa}\varepsilon = \sup_{B_{\sqrt{\kappa}}} u^+ \ge u|_{\partial B_{\sqrt{\kappa}}}, \tag{3.20}$$

that is

$$C_1 = \frac{\sqrt{\kappa}}{e^{-\mu\kappa} - e^{-\mu\kappa^2}}.$$

Furthermore, by a direct computation we can see that

$$\nabla v = -C_1\varepsilon\, 2\mu x e^{-\mu|x|^2} \quad \text{in } B_{\sqrt{\kappa}} \setminus B_\kappa, \tag{3.21}$$

and

$$\Delta_p v(x) = C_1\varepsilon\,(p-1)(2\mu)^2|\nabla v|^{p-2}e^{-\mu|x|^2}\left(|x|^2 - \frac{N+p-2}{2\mu(p-1)}\right),$$

see [21]. Thus

$$v \text{ is } p-\text{superharmonic in } B_{\sqrt{\kappa}} \setminus B_\kappa \tag{3.22}$$

if μ is sufficiently small, say,

$$\mu < \frac{N+p-2}{2\kappa(p-1)}.$$

It is clear that $\min\{u,v\} = u$ on $\partial B_{\sqrt{\kappa}}$, thanks to (3.20), hence by the minimality of u (recall also Subsection 3.4)

$$\tilde{J}(u) \le \tilde{J}(\min\{u,v\}). \tag{3.23}$$

Now we observe that

$$\begin{aligned}\tilde{J}(\min\{u,v\}) &= \int_{B_\kappa} |\nabla \min\{u,v\}|^p + \Lambda\chi_{\{\min\{u,v\}>0\}} \\ &\quad + \int_{B_{\sqrt{\kappa}}\setminus B_\kappa} |\nabla \min\{u,v\}|^p + \Lambda\chi_{\{\min\{u,v\}>0\}} \\ &= \int_{B_\kappa\cap\{u\le 0\}} |\nabla u|^p + \Lambda\chi_{\{u>0\}} \\ &\quad + \int_{B_{\sqrt{\kappa}}\setminus B_\kappa} |\nabla \min\{u,v\}|^p + \Lambda\chi_{\{\min\{u,v\}>0\}},\end{aligned}$$

while

$$\begin{aligned}\tilde{J}(u) &= \int_{B_\kappa\cap\{u\le 0\}} |\nabla u|^p + \Lambda\chi_{\{u>0\}} \\ &\quad + \int_{B_\kappa\cap\{u>0\}} |\nabla u|^p + \Lambda\chi_{\{u>0\}} + \int_{B_{\sqrt{\kappa}}\setminus B_\kappa} |\nabla u|^p + \Lambda\chi_{\{u>0\}}.\end{aligned}$$

Therefore, from (3.23), we have that

$$\begin{aligned}\int_{B_\kappa\cap\{u>0\}} |\nabla u|^p + \Lambda\chi_{\{u>0\}} &\le \int_{B_{\sqrt{\kappa}}\setminus B_\kappa} |\nabla \min\{u,v\}|^p + \Lambda\chi_{\{\min\{u,v\}>0\}} \\ &\quad - \int_{B_{\sqrt{\kappa}}\setminus B_\kappa} |\nabla u|^p + \Lambda\chi_{\{u>0\}} \\ &\le \int_{B_{\sqrt{\kappa}}\setminus B_\kappa} |\nabla \min\{u,v\}|^p - |\nabla u|^p \\ &= \int_{(B_{\sqrt{\kappa}}\setminus B_\kappa)\cap\{u>v\}} |\nabla v|^p - |\nabla u|^p \\ &\le -p\int_{B_{\sqrt{\kappa}}\setminus B_\kappa} |\nabla v|^{p-2}\nabla v\cdot\nabla\max\{u-v,0\} \\ &= -p\int_{B_{\sqrt{\kappa}}\setminus B_\kappa} -\Delta_p v\max\{u-v,0\} + \operatorname{div}(|\nabla v|^{p-2}\nabla v\max\{u-v,0\}) \\ &\le -p\int_{B_{\sqrt{\kappa}}\setminus B_\kappa} \operatorname{div}(|\nabla v|^{p-2}\nabla v\max\{u-v,0\}) \\ &= p\int_{\partial B_\kappa} |\nabla v|^{p-2}\nabla v\cdot\nu\max\{u-v,0\} \\ &= p\int_{\partial B_\kappa} |\nabla v|^{p-2}(\nabla v\cdot\nu)u^+,\end{aligned}$$

where to get the last line we also used the fact that v is a $p-$supersolution in $B_{\sqrt{\kappa}}\setminus B_\kappa$ (recall (3.22)) and (3.20)). Moreover, by (3.21), we have that $|\nabla v| = C_1\varepsilon 2\mu\kappa e^{-\mu\kappa^2} \le C\varepsilon$ on ∂B_κ, for some $C>0$. Thus

$$\int_{B_\kappa\cap\{u>0\}} |\nabla u|^p + \Lambda\chi_{\{u>0\}} \le p(C\varepsilon)^{p-1}\int_{\partial B_\kappa} u^+. \tag{3.24}$$

On the other hand, from trace estimate, Young's inequality and (3.19), we get

$$\begin{aligned}
\int_{\partial B_\kappa} u^+ &\leq C(N,\kappa)\left(\int_{B_\kappa} u^+ + \int_{B_\kappa} |\nabla u^+|\right)\\
&\leq C(N,\kappa)\left(\sup_{B_\kappa} u^+ \int_{B_\kappa} \chi_{\{u>0\}} + \int_{B_\kappa} \frac{1}{p}|\nabla u^+|^p + \frac{1}{p'}\chi_{\{u>0\}}\right)\\
&\leq C(N,\kappa)\left((\varepsilon\sqrt{\kappa} + \frac{1}{p'})\int_{B_\kappa} \chi_{\{u>0\}} + \frac{1}{p}\int_{B_\kappa} |\nabla u^+|^p\right)\\
&\leq C_0 \int_{B_\kappa\cap\{u>0\}} |\nabla u|^p + \Lambda\chi_{\{u>0\}},
\end{aligned} \tag{3.25}$$

where p' is the conjugate of p and

$$C_0 := C(N,\kappa)\left(\frac{\varepsilon\sqrt{\kappa} + 1/p'}{\Lambda} + \frac{1}{p}\right).$$

Thereby, putting together (3.24) and (3.25), we obtain

$$\int_{B_\kappa\cap\{u>0\}} |\nabla u|^p + \Lambda\chi_{\{u>0\}} \leq p(C\varepsilon)^{p-1}\, C_0 \int_{B_\kappa\cap\{u>0\}} |\nabla u|^p + \Lambda\chi_{\{u>0\}},$$

which implies that $u \equiv 0$ in B_κ if ε is small enough. ☐

As a consequence of Proposition 3.5 we have:

Corollary 3.6. *Let u be as in Proposition 3.5. Let $x \in \partial\{u > 0\}$ and $r > 0$ such that $B_r(x) \subset \Omega$. Then*

$$\fint_{B_r(x)} (u^+)^p \geq cr,$$

where c depends only on $\Lambda = \lambda_+^p - \lambda_-^p > 0$.

3.6. **One phase control implies linear growth.** The last technical estimate is very weak and of pointwise nature. It is used in the proof of Theorem 4.2 and serves a preliminary step towards the proof of Theorem A.

Lemma 3.7. *Let u be a bounded local minimizer of (1.1). Let $x_0 \in \partial\{u > 0\}$ and $r > 0$ small such that $B_r(x_0) \subset \Omega$. Assume that $\sup_{B_r(x_0)} u^- \leq C_0 r$ (resp. $\sup_{B_r(x_0)} u^+ \leq C_0 r$), for some constant C_0 depending on x_0.*

Then there exists a constant $\sigma > 0$ such that $\sup_{B_r(x_0)} u^+ \leq \sigma C_0 r$ (resp. $\sup_{B_r(x_0)} u^- \leq \sigma C_0 r$).

Proof. We will show only one of the claims, the other can be proved analogously. Suppose that

$$\sup_{B_r(x_0)} u^- \leq C_0 r \tag{3.26}$$

and we claim that

$$S(k+1) \leq \max\left\{\frac{\sigma C_0}{2^{k+1}}, \frac{1}{2}S(k)\right\}, \tag{3.27}$$

where $S(k) := \sup_{B_{2^{-k}}(x_0)} |u|$, for any $k \in \mathbb{N}$. To prove this, we argue by contradiction and we suppose that (3.27) fails. Then there is a sequence of integers k_j, with $j = 1, 2, \ldots$, such that

$$S(k_j+1) > \max\left\{\frac{j}{2^{k_j+1}}, \frac{1}{2}S(k_j)\right\}. \tag{3.28}$$

Observe that since u is a bounded minimizer, then (3.28) implies that $k_j \to \infty$ as $j \to +\infty$. Also, notice that (3.28) implies that

$$\frac{2^{-k_j}}{S(k_j+1)} \leq \frac{2}{j} \to 0 \quad \text{as } j \to +\infty. \tag{3.29}$$

Now, we introduce the scaled functions $v_j(x) := \frac{u(x_0+2^{-k_j}x)}{S(k_j+1)}$, for any $x \in B_1$. Then, from (3.26) and (3.29), it follows that

$$v_j(0) = 0 \quad \text{and} \quad v_j^-(x) = \frac{u^-(x_0+2^{-k_j}x)}{S(k_j+1)} \leq \frac{2^{-k_j}C_0}{S(k_j+1)} < \frac{2C_0}{j} \to 0 \text{ as } j \to +\infty. \tag{3.30}$$

Also, by (3.16) (used here with $\rho := 2^{-k_j}$ and $S := S(k_j+1)$) we see that v_j is a minimizer of the functional

$$\int_{B_1} |\nabla v_j(x)|^p + \left[\frac{2^{-k_j-1}}{S(k_j+1)}\right]^p \Lambda \chi_{\{v_j>0\}}.$$

Furthermore, it is not difficult to see that (3.28) implies that

$$\sup_{B_1} |v_j| \leq 2, \quad \text{and} \quad \sup_{B_{\frac{1}{2}}} |v_j| = 1. \tag{3.31}$$

Using this and Caccioppoli's inequality, we infer that

$$\int_{B_{\frac{3}{4}}} |\nabla v_j^{\pm}|^p \leq 4^p C(N) \int_{B_1} (v_j^{\pm})^p \leq 2^{3p} C(N),$$

for some $C(N) > 0$, implying that $\|v_j\|_{W^{1,p}(B_{\frac{3}{4}})}$ are uniformly bounded. So using Lemma 3.1 we can extract a converging subsequence such that $v_j \to v_0$ uniformly in $\overline{B_{\frac{3}{4}}}$ and $\nabla v_j \to \nabla v_0$ in $L^q(B_{\frac{3}{4}})$ for any $q > 1$. Moreover, by (3.29),

$$\int_{B_{\frac{3}{4}}} |\nabla v_j(x)|^p + \left[\frac{2^{-k_j-1}}{S(k_j+1)}\right]^p \Lambda \chi_{\{v_j>0\}} \to \int_{B_{\frac{3}{4}}} |\nabla v_0(x)|^p, \quad \text{as } j \to +\infty.$$

This, (3.30) and (3.31) give that

$$\Delta_p v_0(x) = 0, \quad v_0(x) \geq 0 \text{ if } x \in B_{\frac{3}{4}}, \quad v_0(0) = 0, \quad \text{and } \sup_{B_{\frac{1}{2}}} v_0 = 1$$

which is in contradiction with the strong minimum principle. This shows (3.27) and finishes the proof. □

4. Viscosity solutions

In order to exploit the regularity theory of free boundary developed for the viscosity solutions in [23, 24] we shall prove that any $W^{1,p}$ minimizer of J is also viscosity solution, as opposed to Definition 2.4 in [9]. For this, we recall that $\Omega^+(u) = \{u > 0\}$ and $\Omega^-(u) = \{u < 0\}$. Moreover, if the free boundary is C^1 smooth then

$$G(u_\nu^+, u_\nu^-) := (u_\nu^+)^p - (u_\nu^-)^p - \Lambda_0 \tag{4.1}$$

is the flux balance across the free boundary, where u_ν^+ and u_ν^- are the normal derivatives in the inward direction to $\partial\Omega^+(u)$ and $\partial\Omega^-(u)$, respectively (recall that $\Lambda_0 = \frac{\Lambda}{p-1} = \frac{\lambda_+^p - \lambda_-^p}{p-1}$ is the Bernoulli constant).

Definition 4.1. *Let Ω be a bounded domain of $\mathbb{R}^N$ and let u be a continuous function in Ω. We say that u is a viscosity solution in Ω if*

i) $\Delta_p u = 0$ *in* $\Omega^+(u)$ *and* $\Omega^-(u)$,

ii) *along the free boundary Γ, u satisfies the free boundary condition, in the sense that:*

a) *if at $x_0 \in \Gamma$ there exists a ball $B \subset \Omega^+(u)$ such that $x_0 \in \partial B$ and*

$$u^+(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|), \quad \text{for } x \in B, \tag{4.2}$$

$$u^-(x) \leq \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|), \quad \text{for } x \in B^c, \tag{4.3}$$

for some $\alpha > 0$ and $\beta \geq 0$, with equality along every non-tangential domain, then the free boundary condition is satisfied

$$G(\alpha, \beta) = 0,$$

b) *if at $x_0 \in \Gamma$ there exists a ball $B \subset \Omega^-(u)$ such that $x_0 \in \partial B$ and*

$$u^-(x) \geq \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|), \quad \text{for } x \in B,$$

$$u^+(x) \leq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|), \quad \text{for } x \in \partial B,$$

for some $\alpha \geq 0$ and $\beta > 0$, with equality along every non-tangential domain, then

$$G(\alpha, \beta) = 0.$$

The main result of this section is the following:

Theorem 4.2. *Let $u \in W^{1,p}(\Omega)$ be a minimizer of* (1.1). *Then u is a viscosity solution in Ω in the sense of Definition 4.1.*

The proof of Theorem 4.2, will follow from Lemma 4.3 below. It is a generalization of Lemma 11.17 in [9] to any p (see also the appendix in [12], where the authors deal with the one-phase problem in the half ball.) We postpone the proof of Lemma 4.3 to Appendix A.

Lemma 4.3. *Let $0 \leq u \in W^{1,p}(\Omega)$ be a solution of $\Delta_p u = 0$ in Ω and $x_0 \in \partial\Omega$. Suppose that u continuously vanishes on $\partial\Omega \cap B_1(x_0)$. Then*

a) *if there exists a ball $B \subset \Omega$ touching $\partial\Omega$ at x_0, then either u grows faster than any linear function at x_0, or there exists a constant $\alpha > 0$ such that*

$$u(x) \geq \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \quad \text{in } B, \tag{4.4}$$

where ν is the unit normal to ∂B at x_0, inward to Ω. Moreover, equality holds in (4.4) *in any non-tangential domain.*

b) *if there exists a ball $B \subset \Omega^c$ touching $\partial\Omega$ at x_0, then there exists a constant $\beta \geq 0$ such that*

$$u(x) \leq \beta\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \quad \text{in } B^c, \tag{4.5}$$

with equality in any non-tangential domain.

With this, we are able to prove Theorem 4.2.

Proof of Theorem 4.2. First we observe that i) in Definition 4.1 is satisfied, thanks to P.1 in Proposition 3.4.

To prove ii), we let $x_0 \in \Gamma \cap B$, $B \subset \{u > 0\}$ be a ball touching Γ at x_0 and ν be the unit vector at x_0 pointing to the centre of B. We want to show that (4.2) and (4.3) are satisfied for some $\alpha > 0$ and $\beta \geq 0$, with equality in every non-tangential domain.

Notice that β is finite, thanks to Lemma 4.3 (in particular, the statement b) applied to u^-). This follows from a standard barrier argument as one compares u^- with

$$b(x) = \sup_{B_{2r}(y_0)} u^- \frac{\Phi(|x - y_0|) - \Phi(r)}{\Phi(2r) - \Phi(r)}, \quad x \in B_{2r}(y_0) \setminus B_r(y_0)$$

where $\Phi(t) = t^{\frac{p-N}{p-1}}$, r is the radius and y_0 the centre of B.

Thus α is finite too, according to Lemma 3.7, that is

$$\alpha < \infty, \quad \beta < \infty. \tag{4.6}$$

Recall that, using the notation in [9, 23, 24], the free boundary condition takes the form (4.1)

$$G(\alpha, \beta) := \alpha^p - \beta^p - \Lambda_0.$$

Therefore it is enough to show that

$$\alpha^p - \beta^p = \Lambda_0. \tag{4.7}$$

For this, we first consider the case $\beta = 0$, i.e. when u^- is degenerate. We define the scaled function at x_0

$$u_\rho(x) := \frac{u(x_0 + \rho x)}{\rho}, \quad 0 < \rho < \operatorname{dist}(x_0, \partial\Omega).$$

Since x_0 is a non-flat point of free boundary then it follows from (4.6) that for any sequence $\rho_j \to 0$ as $j \to +\infty$ there is a subsequence $\rho_{j(k)} \to 0$ such that $u_{\rho_{j(k)}}$ converges to some u_0. Moreover, owing to Lemma 4.3, in a non-tangential domain we have that

$$u_\rho(x) = \alpha\langle x, \nu\rangle^+ + \frac{o(\rho|x|)}{\rho} \to \alpha\langle x, \nu\rangle^+ \quad \text{as } \rho \to 0.$$

Without loss of generality, we may assume that $\nu = e_1$. Thus, after blowing-up, we have that $u_0 = \alpha x_1^+$ in a cone $K_0 := \{x = (x_1, x') \in \mathbb{R}\times\mathbb{R}^{N-1}, s.t.\ x_1 \geq |x'| \cos\theta\}$ for some $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$. Notice that $u_0 > 0$ in $K_0 \cap (B_2 \setminus B_1)$. Also, $\Delta_p u_0 = 0$ in $K_0 \cap (B_2 \setminus B_1)$. Then, by the Unique Continuation Theorem (see Proposition 5.1 in [19]) we get that $u_0 = \alpha x_1^+$ in $\mathbb{R}^N$. In turn, this implies that the free boundary condition is satisfied in the classical sense on the hyperplane $\{x_1 = 0\}$. That is, $|\nabla u_0|^p = \Lambda_0$ on $\{x_1 = 0\}$, and so $\alpha^p = \Lambda_0$ on $\{x_1 = 0\}$. Hence (4.7) is satisfied in the case $\beta = 0$.

Suppose now that $\beta > 0$, namely u^- is non-degenerate. Reasoning as above and blowing-up, we can prove that $u_0^+ = \alpha x_1^+$. It remains to show that $u_0^- = \beta x_1^-$. To do this, we set $\Gamma_0 := \partial\{u_0 > 0\}$, that is Γ_0 is the free boundary of the blow-up u_0. We take $z \in \{x_1 = 0\}$, $z \neq 0$, and we take the ball $B_r(z)$ for some $0 < r < |z|$, see Figure 1.

There are three possibilities:

Case 1) u_0 vanishes only on $B_r(z) \cap \{x_1 = 0\}$ and $u_0 > 0$ in $B_r(z) \cap \{x_1 < 0\}$,
Case 2) u_0 vanishes only on $B_r(z) \cap \{x_1 = 0\}$ and $u_0 < 0$ in $B_r(z) \cap \{x_1 < 0\}$,
Case 3) u_0 vanishes in $B_r(z) \cap \{x_1 \leq 0\}$.

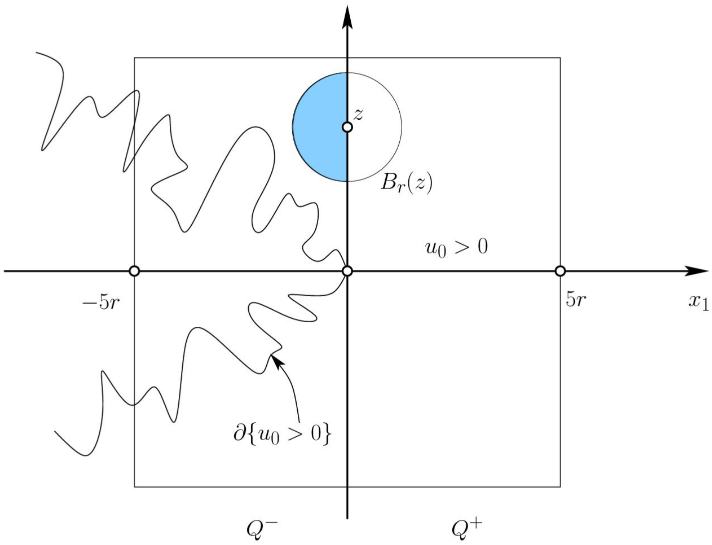


FIGURE 1. In the coloured half of $B_r(z)$ either $u_0 < 0$ or $u_0 \equiv 0$.

Notice that **Case 1)** cannot occur, because it would imply that we deal with a one-phase problem in $B_r(z)$ and the $\mathcal{H}^N-$density estimate for the zero set would be violated (see Theorem 4.4 in [11]).

Consider now **Case 2)**, and observe that on the hyperplane $\{x_1 = 0\}$ the free boundary condition is satisfied in classical sense:

$$|\nabla u_0^-|^p = \alpha^p - \Lambda_0 =: \gamma^p$$

in $B_r(z)$. In particular,

$$|\nabla u_0^-| = \gamma > 0 \quad \text{on } \{x_1 = 0\} \cap B_r(z). \tag{4.8}$$

Define

$$\tilde{u}_0 := \begin{cases} u_0^- & \text{in } B_r(z) \cap \{x_1 < 0\}, \\ \gamma x_1 & \text{in } B_r(z) \cap \{x_1 > 0\}. \end{cases}$$

We claim that

$$\Delta_p \tilde{u} = 0 \quad \text{in } B_{r/2}(z). \tag{4.9}$$

Indeed, $\tilde{u}$ is p-harmonic in $B_r(z) \cap \{x_1 > 0\}$. Moreover, (4.8) yields that $u_0^- \in C^2(\overline{B_{r/2}(z) \cap \{x_1 < 0\}})$, therefore we have that $\Delta_p \tilde{u} = 0$ pointwise in $\overline{B_{r/2}(z) \cap \{x_1 < 0\}}$, and so (4.9) follows.

Hence, from (4.8), (4.9) and the Unique Continuation Theorem [19] we obtain that u_0^- must be a linear function in $B_{r/2}(z) \cap \{x_1 < 0\}$. Then, Proposition 5.1 in [19] implies that u_0^- is a linear function in $\{x_1 < 0\}$. Thus the free boundary condition is satisfied in the classical sense on the plane $\{x_1 = 0\}$ including the origin, and this proves equality in (4.7) in Case 2).

Now we deal with **Case 3)**. We consider a cube $Q = (-5r, 5r) \times (-5r, 5r)$ centered at the origin such that $B_r(z) \subset Q$, and we set $Q^- := Q \cap \{x_1 < 0\}$. Notice that

$$u_0 \le 0 \quad \text{in } Q^-. \tag{4.10}$$

In particular, $u_0 \leq 0$ on ∂Q^-. According to the remark in Subsection 3.4, u_0 is a minimizer in Q^- of the functional

$$\tilde{J}(u) = \int_{Q^-} |\nabla u|^p + \Lambda \chi_{\{u>0\}} = \int_{Q^-} |\nabla u|^p.$$

Therefore u_0 is p-harmonic in Q^-. By maximum principle, u_0 cannot achieve its maximum inside Q^-. This and (4.10) imply that $u_0 < 0$ in Q^-, and so the free boundary coincides with $\{x_1 = 0\}$.

This concludes the proof of ii)-a) in Definition 4.1. Similarly, one can also prove ii)-b). Hence, u is a viscosity solution, and the desired result follows. □

5. On $\varepsilon-$monotonicity of u and slab flatness of $\partial\{u > 0\}$

One of the main free boundary regularity theorems for viscosity solutions is formulated in terms of the $\varepsilon-$monotonicity of u. More precisely, we have:

Definition 5.1. *We say that u is $\varepsilon-$monotone if there are a unit vector e and an angle θ_0 with $\theta_0 > \frac{\pi}{4}$ (say) and $\varepsilon > 0$ (small) such that, for every $\varepsilon' \geq \varepsilon$,*

$$\sup_{B_{\varepsilon' \sin\theta_0}(x)} u(y - \varepsilon' e) \leq u(x). \tag{5.1}$$

We denote by $\Gamma(\theta_0, e)$ the cone with axis e and opening θ_0.

Definition 5.2. *We say that u is $\varepsilon-$monotone in the cone $\Gamma(\theta_0, \varepsilon)$ if it is $\varepsilon-$monotone in any direction $\tau \in \Gamma(\theta_0, \varepsilon)$.*

One can interpret the $\varepsilon-$monotonicity of u as closeness of the free boundary to a Lipschitz graph with Lipschitz constant sufficiently close to 1 if we leave the free boundary in directions e at distance ε and higher. The exact value of the Lipschitz constant is given by $\left(\tan\frac{\theta_0}{2}\right)^{-1}$. Then the ellipticity propagates to the free boundary via Harnack's inequality giving that Γ is Lipschitz. Furthermore, Lipschitz free boundaries are, in fact, $C^{1,\alpha}$ regular.

For $p = 2$ this theory was founded by L. Caffarelli, see [6, 7, 8]. Recently J. Lewis and K. Nyström proved that this theory is valid for all $p > 1$, see [23, 24]. In fact, their argument does not require u to be Lipschitz.

For viscosity solutions we replace the $\varepsilon-$monotonicity with the slab flatness measuring the thickness of $\partial\{u > 0\} \cap B_r(x)$ in terms of the quantity $h(x, r)$ introduced in (2.3). In other words, $h(x, r)$ measures how close the free boundary is to a pair of parallel planes in a ball $B_r(x)$ with $x \in \Gamma$. Clearly, planes are Lipschitz graphs in the direction of the normal, therefore the slab flatness of Γ is a particular case of $\varepsilon-$monotonicity of u.

Hence, under h_0-flatness of the free boundary we can reformulate the regularity theory "flatness implies $C^{1,\alpha}$" as follows:

Theorem 5.3. *Suppose that $B_r(x_0) \subset \Omega$ with $x_0 \in \partial\{u > 0\}$. Then there exists $h > 0$ such that if $\Gamma \cap B_r(x_0) \subset \{x \in \mathbb{R}^N : -hr < (x - x_0) \cdot \nu < hr\}$ then $\Gamma \cap B_{r/2}(x_0)$ is locally $C^{1,\alpha}$ in the direction of ν, for some $\alpha \in (0, 1)$.*

6. Linear growth vs flatness: Proofs of Theorems A and A′

6.1. **Dyadic scaling.** We first discuss a preliminary result, that we will use for the proof of Theorem A.

Proposition 6.1. *Let u be a local minimizer of J and $x_0 \in \Gamma \cap B_1 \subset \Omega$. For any $k \in \mathbb{N}$, set*

$$S(k,u) := \sup_{B^+_{2^{-k}}(x_0)} |u|.$$

If $h_0 > 0$ is fixed and $h\left(x_0, \frac{1}{2^k}\right) \geq \frac{h_0}{2^{k+1}}$ for some k, then

$$S(k+1,u) \leq \max\left\{\frac{L2^{-k}}{2}, \frac{S(k,u)}{2}, \dots, \frac{S(k-m,u)}{2^{m+1}}, \dots, \frac{S(0,u)}{2^{k+1}}\right\}, \tag{6.1}$$

for some positive constant L, that is independent of x_0 and k.

Otherwise if $h\left(x_0, \frac{1}{2^k}\right) < \frac{h_0}{2^{k+1}}$ for some k, then $\Gamma \cap B_{2^{-(k+1)}}$ is a $C^{1,\alpha}$ smooth surface, for some $\alpha \in (0,1)$.

Proof. We first deal with the case $h\left(x_0, \frac{1}{2^k}\right) \geq \frac{h_0}{2^{k+1}}$. In order to prove (6.1), we use a contradiction argument discussed in [22]. Hence, we suppose that (6.1) fails, that is there exist integers k_j, $j = 1, 2, \dots$, local minimizers u_j and points $x_j \in \Gamma_j \cap B_1$ such that

$$h\left(x_j, \frac{1}{2^{k_j}}\right) \geq \frac{h_0}{2^{k_j+1}} \tag{6.2}$$

and

$$S(k_j+1,u_j) > \max\left\{\frac{j2^{-k_j}}{2}, \frac{S(k_j,u_j)}{2}, \dots, \frac{S(k_j-m,u_j)}{2^{m+1}}, \dots, \frac{S(0,u_j)}{2^{k_j+1}}\right\}. \tag{6.3}$$

Since u_j is a local minimizer of J in B_1 and $u_j(x_j) = 0$, then u_j is bounded (see Theorem 1 in [22]). Namely, there exists a positive constant M, that is independent of j, such that $S(k_j + 1, u_j) \leq M$. Therefore, from (6.3) we have that $M \geq j2^{-k_j}/2$, which implies that $2^{k_j} \geq j/(2M)$. Hence, k_j tends to $+\infty$ when $j \to +\infty$.

We set

$$\sigma_j := \frac{2^{-k_j}}{S(k_j+1,u_j)}. \tag{6.4}$$

Using (6.3) once more, we see that

$$\sigma_j < \frac{2}{j} \to 0 \ \text{ as } j \to +\infty. \tag{6.5}$$

For any j, we now define the function

$$v_j(x) := \frac{u_j(x_j + 2^{-k_j}x)}{S(k_j+1,u_j)}. \tag{6.6}$$

Then, by construction,

$$\sup_{B_{1/2}} |v_j| = 1. \tag{6.7}$$

Furthermore, from (6.3) we have that

$$1 > \max\left\{\frac{j2^{-k_j}}{2S(k_j+1,u_j)}, \frac{1}{2}\sup_{B_1}|v_j|, \dots, \frac{1}{2^{m+1}}\sup_{B_{2^m}}|v_j|, \dots, \frac{1}{2^{k_j+1}}\sup_{B_{2^{k_j+1}}}|v_j|\right\},$$

which in turn implies that

$$\sup_{B_{2^m}} |v_j| \le 2^{m+1}, \quad \text{for any } m < 2^{k_j}. \tag{6.8}$$

Finally, since $u_j(x_j) = 0$, we have that

$$v_j(0) = 0. \tag{6.9}$$

Notice that v_j is a minimizer (according to its own boundary values) of the scaled functional

$$\widehat{J}(v) := \int_{B_R} |\nabla v|^p + \sigma_j^p \lambda(v), \tag{6.10}$$

for $0 < R < 2^{k_j}$ and j large. Indeed, from (6.6) and an easy computation, we get

$$\nabla v_j(x) = \frac{2^{-k_j}}{S(k_j+1, u_j)} \nabla u_j(x_j + 2^{-k_j} x).$$

Hence, by the change of variable $y = x_j + 2^{-k_j} x$ and recalling (6.4),

$$\begin{aligned}
\widehat{J}(v_j) &= \int_{B_R} |\nabla v_j(x)|^p + \sigma_j^p \lambda(v)\, dx \\
&= \int_{B_R} \frac{2^{-pk_j}}{S(k_j+1, u_j)^p} |\nabla u_j(x_j + 2^{-k_j} x)|^p + \sigma_j^p \lambda(u_j(x_j + 2^{-k_j} x))\, dx \\
&= \sigma_j^p \, 2^{nk_j} \int_{B_{R2^{-k_j}}(x_j)} |\nabla u_j(y)|^p + \lambda(u_j)\, dy.
\end{aligned}$$

Since u_j is a minimizer for J, the last formula implies that v_j is a minimizer for $\widehat{J}$. Hence, from Lemma 3.1 we obtain that for any $q > 1$ and $0 < R < 2^{k_j}$ there exists a constant $C = C(R, q) > 0$ independent of j such that

$$\max\{\|v_j\|_{C^\alpha(B_R)}, \|\nabla v_j\|_{L^q(B_R)}\} \le C,$$

for some $\alpha \in (0,1)$. Therefore, by a standard compactness argument, we have that, up to a subsequence,

(6.11) v_j converges to some function v as $j \to +\infty$ in $W^{1,q}(B_R) \cap C^\alpha(B_R)$ for any fixed R.

From (6.7), (6.8) and (6.9) we obtain that

$$\sup_{B_{1/2}} |v| = 1, \quad \sup_{B_{2^m}} |v| \le 2^{m+1} \quad \text{and} \quad v(0) = 0.$$

We claim that

(6.12) v is a minimizer for the functional $\mathcal{J}(v) := \int_{B_R} |\nabla v|^p$.

For this, notice that for any $\varphi \in C_0^\infty(B_R)$

$$\int_{B_R} |\nabla v_j|^p + \sigma_j^p \lambda(v_j) \le \int_{B_R} |\nabla (v_j + \varphi)|^p + \sigma_j^p \lambda(v_j + \varphi), \tag{6.13}$$

because v_j is a minimizer for $\widehat{J}$ defined in (6.10). By taking $q > p$ in (6.11), we have that

$$\begin{aligned}
&\int_{B_R} |\nabla v_j|^p \to \int_{B_R} |\nabla v|^p \\
\text{and} \qquad &\int_{B_R} |\nabla (v_j + \varphi)|^p \to \int_{B_R} |\nabla (v + \varphi)|^p
\end{aligned}$$

as $j \to +\infty$. Moreover, from (6.5) we obtain

$$\int_{B_R} \sigma_j^p \lambda(v_j) \to 0 \quad \text{and} \quad \int_{B_R} \sigma_j^p \lambda(v_j + \varphi) \to 0$$

as $j \to +\infty$. Thus, sending $j \to +\infty$ in (6.13) and using these observations, we get

$$\int_{B_R} |\nabla v|^p \leq \int_{B_R} |\nabla (v + \varphi)|^p$$

for any $\varphi \in C_0^\infty(B_R)$. This implies (6.12).

Hence, from Liouville's Theorem (see Theorem 3.3) we deduce that v must be a linear function in $\mathbb{R}^N$. Without loss of generality we can take $v(x) = Cx_1$ for some positive constant C.

On the other hand, (6.2) implies that the following inequality holds true for the function v_j:

$$h(0,1) \geq \frac{h_0}{2}.$$

By the uniform convergence in (6.11), we have that for any $\varepsilon > 0$ there is j_0 such that $|Cx_1 - v_j(x)| < \varepsilon$ whenever $j > j_0$. Since $\partial\{v_j > 0\}$ is $h_0/2$ thick in B_1 it follows that there is $y_j \in \partial\{v_j > 0\} \cap B_1$ such that $y_j = e_1 h_0/4 + t_j e'$, for some $t_j \in \mathbb{R}$, where e_1 is the unit direction of x_1 axis and $e' \perp e_1$. Then we have that $|C\frac{h_0}{4} - 0| = |v(y_j) - v_j(y_j)| < \varepsilon$, which is a contradiction if ε is small. This finishes the proof of (6.1).

If $h(x_0, \frac{1}{2^k}) < \frac{h_0}{2^{k+1}}$ for some k, we use Theorem 4.2 to obtain that u is also a viscosity solution in the sense of Definition 4.1. Therefore, we can apply the regularity result in Theorem 5.3, thus obtaining the desired claim. This concludes the proof of Proposition 6.1. □

6.2. **Proof of Theorem A.** With the aid of Proposition 6.1 we now complete the proof of Theorem A.

Proof of Theorem A. The argument in Proposition 6.1 shows that either there are finitely many integers k such that

$$h\left(x_0, \frac{1}{2^k}\right) \geq \frac{h_0}{2^{k+1}} \tag{6.14}$$

and

$$S(k+1, u) \leq \max\left\{\frac{L2^{-k}}{2}, \frac{S(k,u)}{2}, \dots, \frac{S(k-m,u)}{2^{m+1}}, \dots, \frac{S(0,u)}{2^{k+1}}\right\}, \tag{6.15}$$

or there are infinitely many k such that (6.14) and (6.15) hold true.

In the first case, there exists k_0 such that $h\left(x_0, \frac{1}{2^{k_0}}\right) < \frac{h_0}{2^{k_0+1}}$, and so $\Gamma \cap B_{2^{-(k_0+1)}}$ is a $C^{1,\alpha}$ smooth surface. In the second case, we have linear growth of u at the free boundary point x_0 where the flatness does not improve.

Suppose now that we are given $r > 0$. Then, either $h(x_0, r) < \frac{h_0}{2} r$ or $h(x_0, r) \geq \frac{h_0}{2} r$. In the first case, we obtain that Γ is a $C^{1,\alpha}$-surface. In the second case we argue as follows: there exists $k \in \mathbb{N}$ such that

$$\frac{1}{2^{k+1}} \leq r \leq \frac{1}{2^k}.$$

Hence, by the definition of h given in (2.3), we have that

$$h\left(x_0, \frac{1}{2^{k+1}}\right) \geq h(x_0, r) \geq \frac{h_0}{2} r \geq \frac{h_0}{2} \frac{1}{2^{k+1}}.$$

This means that we are in the position to apply Proposition 6.1, that implies linear growth of u at the level $r/2$. $\square$

A refinement of Theorem A is given by the following:

Corollary 6.2. *Let h_0 be the constant given in Theorem A. Then, if $r \in [2^{-k-1}, 2^{-k})$ and $h(x_0, r) \geq \frac{h_0}{2} r$, we have that*

$$\sup_{B_{\frac{r}{2}}(x_0)} |u| \leq 2Lr,$$

where L is the constant given by Theorem A.

6.3. **Alt-Caffarelli-Friedman functional.** Here we introduce a functional that is a generalization to any $p > 1$ of the one introduced by Alt, Caffarelli and Friedman in the case $p = 2$, and we show that this functional is bounded at non-flat free boundary points, thanks to the linear growth ensured by Theorem A.

For this, we let $u = u^+ - u^-$, where $u^+ := \max\{0, u\}$ and $u^- := -\min\{0, u\}$. We define the functional

$$\varphi_p(r, u, x_0) := \frac{1}{r^4} \int_{B_r(x_0)} \frac{|\nabla u^+|^p}{|x - x_0|^{N-2}} \int_{B_r(x_0)} \frac{|\nabla u^-|^p}{|x - x_0|^{N-2}}$$

where $x_0 \in \partial\{u > 0\}$ and $r > 0$ is such that $B_r(x_0) \subset \Omega$.

Precisely, we show the following:

Corollary 6.3. *Let h_0 be fixed, $D \Subset \Omega$ be a subdomain and $x_0 \in \Gamma \cap D$ be such that $h(r, x_0) \geq \frac{h_0}{2} r$.*

Then there exist $M > 0, r_0 > 0$ depending only on $N, p, h_0, \sup_\Omega |u|, \Lambda$ and $\operatorname{dist}(D, \partial\Omega)$ such that

$$\varphi_p(r, u, x_0) \leq \frac{M^2 N^2}{4}, \quad \forall r \leq r_0.$$

Proof. Since $u^\pm$ is nonnegative $p-$subsolution (recall P.1 in Proposition 3.4), we can apply Caccioppoli's inequality, obtaining that

$$\int_{B_\rho(x_0)} |\nabla u^\pm|^p \leq \frac{C}{\rho^p} \int_{B_{2\rho}(x_0)} (u^\pm)^p.$$

From this and Corollary 6.2 we have that

$$\int_{B_\rho(x_0)} |\nabla u^\pm|^p \leq M \rho^N,$$

for some $M > 0$. Hence, using Fubini's Theorem, we have

$$\begin{aligned}
\int_{B_r(x_0)} \frac{|\nabla u^\pm|^p}{|x - x_0|^{N-2}} &= \int_0^r \frac{1}{\rho^{N-2}} \int_{\partial B_\rho(x_0)} |\nabla u^\pm|^p \\
&= \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla u^\pm|^p + (N-2) \int_0^r \frac{1}{\rho^{N-1}} \int_{B_\rho(x_0)} |\nabla u^\pm|^p \\
&\leq \frac{MN}{2} r^2,
\end{aligned}$$

which implies the desired result. $\square$

Remark 6.4. *In [15] we prove the converse statement in some sense. More precisely we show that if $N = 2$ and $p > 2$ is close to 2 then $\varphi_p(r, u, x_0)$ is discrete monotone.*

7. Partial Regularity: Proof of Theorem B

In this section we introduce the set-up in order to prove Theorem B. For this, we recall the notation introduced in Section 2 (recall in particular Definition 2.1 and formula (2.4)). We first show that $\Delta_p u^+$ is Radon measure.

Lemma 7.1. *Let u be a local minimizer of (1.1). Then, the following statements hold true.*

- *$\Delta_p u^+$ is a Radon measure and, for any $x \in \Gamma := \partial\{u > 0\}$ and $r > 0$ such that $B_{2r}(x) \subset \Omega$, there holds*

$$\int_{B_r(x)} \Delta_p u^+ \leq \frac{1}{r} \int_{B_{2r}(x)} |\nabla u^+|^{p-1}. \tag{7.1}$$

- *For a given subdomain $D \Subset \Omega$ there is $r_0 > 0$ such that*

$$\int_{B_r(x)} \Delta_p u^+ \geq C r^{N-1}, \qquad \text{for any } r < r_0, \tag{7.2}$$

for all $x \in \mathcal{N} \cap D$, where $C > 0$ depends on Λ, N, p, $\operatorname{dist}(D, \partial\Omega)$ and L (given by Theorem A).

- *For each $x \in \mathcal{F}$ there is $r(x) > 0$ such that*

$$\int_{B_r(x)} \Delta_p u^+ \geq C r^{N-1}, \qquad \text{for any } r < r(x), \text{ with } B_{r(x)}(x) \subset \Omega, \tag{7.3}$$

for some $C > 0$ that depends on Λ, N, p, $\operatorname{dist}(D, \partial\Omega)$ and x.

Proof. We first show (7.1). For this, we take for simplicity $x = 0$. Observe that by P.1 in Proposition 3.4 we have that $\Delta_p u^+ \geq 0$ in the sense of distributions. Also, for any $\rho \in (r, 2r)$,

$$\int_{B_\rho} \Delta_p u^+ = \int_{\partial B_\rho} |\nabla u^+|^{p-2} \partial_\nu u^+.$$

Therefore, integrating both sides of the last identity over the interval $(r, 2r)$ with respect to ρ, we infer that

$$\begin{aligned} r \int_{B_r} \Delta_p u^+ &\leq \int_r^{2r} \int_{B_\rho} \Delta_p u^+ = \int_r^{2r} \int_{\partial B_\rho} |\nabla u^+|^{p-2} \partial_\nu u^+ \\ &= \int_{B_{2r} \setminus B_r} |\nabla u^+|^{p-2} \nabla u^+ \cdot \frac{x}{|x|} \\ &\leq \int_{B_{2r}} |\nabla u^+|^{p-1}. \end{aligned}$$

This proves (7.1).

To prove (7.2) we argue towards a contradiction. So, for any $j = 1, 2, \ldots$, we let $x_j \in \mathcal{N}$ and $r_j > 0$ such that

$$\int_{B_{r_j}(x_j)} \Delta_p u^+ < \frac{r_j^{N-1}}{j}. \tag{7.4}$$

We also introduce $v_j(x) := \frac{u(x_j + r_j x)}{r_j}$.

Since $x_j \in \mathcal{N}$, it follows from Theorem A that u has uniform linear growth at x_j. This property translates to the scalings of v at x_j giving uniform linear growth for the functions v_j at the origin, i.e. $|v_j(x)| \leq L|x|$ where L is the constant in Theorem A.

Notice that v_j is a minimizer of (1.1), so it is locally C^α, for some $\alpha \in (0,1)$, thanks to Lemma 3.1. Hence $\{v_j\}$ is uniformly bounded in $C^{1,\alpha}$, and so is $\|\nabla v_j\|_{L^p(B_M)}$ for any fixed $M > 0$, thanks to Caccioppoli's inequality. Therefore, we can extract a subsequence $\{r_{j(m)}\}$ such that $v_{j(m)} \to v_0$ as $m \to +\infty$ and v_0 is a minimizer of J in B_2. Moreover, by (7.4),

$$\int_{B_1} \Delta_p v_0^+ = 0,$$

with $v_0^+(0) = 0$. As a consequence, v_0^+ vanishes identically in B_1, by the minimum principle for the $p-$harmonic functions. On the other hand, from Corollary 3.6 we have that $\sup_{B_{\frac{1}{2}}} v_0^+ \geq \frac{c}{2}$, and this gives a contradiction. Thus the proof of (7.2) is finished as well.

The proof of the non-uniform estimate (7.3) follows from a similar argument, by replacing L with a constant $C(x)$ depending on $\nabla u^+(x)$ and $\nabla u^-(x)$. □

As a consequence of Lemma 7.1, we obtain the first part of Theorem B. More precisely:

Corollary 7.2. *Let $R > 0$ be such that $B_R \subset \Omega$. Then $\mathcal{H}^{N-1}(\partial\{u > 0\} \cap B_R) < \infty$.*

Proof. It follows from (7.2) and (7.3) that for each $x \in \Gamma \cap B_R$ there is $r(x) > 0$ such that

$$\int_{B_r(x)} \Delta_p u^+ \geq C r^{N-1}, \quad \text{whenever} \quad r < r(x). \tag{7.5}$$

Thus $\cup_{x\in\Gamma\cap B_R} B_{r(x)}(x)$ is a Besicovitch type covering of $\Gamma \cap B_R$. Applying Besicovitch's Covering Lemma, we have that there is a subcovering $\mathcal{F} = \bigcup_{k=1}^{m(N)} \mathcal{G}_k$ of balls $B_i := B_{r(x_i)}(x_i)$ such that $\sum_i \chi_{B_i} \leq A$ for some dimensional constant $A > 0$ and

$$\Gamma \cap B_R \subset \bigcup_{k=1}^{m(N)} \bigcup_{B_i \in \mathcal{G}_k} B_i,$$

where the balls B_i in each $\mathcal{G}_k$ are disjoint and $\mathcal{G}_k$ are countable.

Now we take a small number $\delta > 0$, and we observe that if $r(x) > \delta$ then (7.5) holds for any $r < \delta$. Hence, without loss of generality, we take $r(x) < \delta$ for any $x \in \Gamma \cap B_R$.

Therefore, using (7.5),

$$\begin{aligned} C \sum_{B_i \in \mathcal{F}} r_i^{N-1} &\leq \sum_{B_i \in \mathcal{F}} \int_{B_i} \Delta_p u^+ \\ &= \sum_{k=1}^{m(N)} \sum_{B_i \in \mathcal{G}_k} \int_{B_i} \Delta_p u^+ \\ &\leq A\, m(N) \int_{B_{8\delta}(\Gamma \cap B_R)} \Delta_p u^+, \end{aligned}$$

where $B_{8\delta}(\Gamma \cap B_R)$ is the 8δ neighbourhood of $\Gamma \cap B_R$. Thus, choosing a finite covering of $B_{8\delta}(\Gamma \cap B_R)$ with balls $B_{R_0}(z_j)$, with $j = 1, \dots, \ell$, such that $B_{2R_0}(z_i) \subset \Omega$ and $B_{8\delta}(\Gamma \cap B_R) \subset \cup_{j=1}^{\ell} B_{R_0}(z_j)$ and using (7.1), we have that

$$\mathcal{H}^{N-1}_\delta(\Gamma \cap B_R) \leq \frac{A}{C 2^{N-1}} \frac{1}{R_0} \sum_{j=1}^{\ell} \int_{B_{2R_0}(z_j)} |\nabla u^+|^{p-1} < +\infty,$$

and letting $\delta \to 0$ we arrive at the desired result. □

We end this section by the following density type estimate to be used in the final stage of the proof of Theorem B.

Lemma 7.3. *For any subdomain $D \Subset \Omega$ there is a positive constant $c \in (0,1)$ depending on $N, p, \Lambda, \sup_\Omega |u|$ and $\operatorname{dist}(D, \partial\Omega)$ such that*

$$\liminf_{r\to 0} \frac{|\{u \le 0\} \cap B_r(x_0)|}{|B_r(x_0)|} \ge c, \qquad \text{for any } x_0 \in D \cap \partial\{u > 0\}. \tag{7.6}$$

Proof. Notice that if $x_0 \in \mathcal{F} \cap D$ then (7.6) holds true with $c = 1/2$. So we focus on the case in which $x_0 \in \mathcal{N} \cap D$.

We fix $r > 0$ such that $B_r(x_0) \subset \Omega$, and we take a function v_r that is $p-$harmonic in $B_r(x_0)$ and such that $u = v_r$ on $\partial B_r(x_0)$. Then, reasoning as at the beginning of the proof of Lemma 3.1 (in particular, using (3.1), (3.2), (3.3) and (3.4)), we have that there exists a tame constant $\bar{c} > 0$ such that

$$\begin{aligned}\bar{c}\int_{B_r(x_0)} |V(\nabla u) - V(\nabla v_r)|^2 &\le \int_{B_r(x_0)} |\nabla u|^p - |\nabla v|^p \\ &\le \int_{B_r(x_0)} \Lambda(\chi_{\{v_r>0\}} - \chi_{\{u>0\}}) \\ &\le \int_{B_r(x_0)} \Lambda\chi_{\{u\le 0\}}.\end{aligned} \tag{7.7}$$

Now we claim that there is a constant $\Theta > 0$ independent of r such that

$$v_r(x_0) \ge \Theta r \quad \text{and} \quad \int_{B_r(x_0)} |V(\nabla u) - V(\nabla v_r)|^2 \ge \frac{\Theta}{r^p}\int_{B_r(x_0)} |u - v_r|^p. \tag{7.8}$$

Notice that by comparison principle it follows that $v_r(x_0) \ge u(x_0) = 0$. We prove the first inequality in (7.8) using a contradiction argument based on compactness, the second one can be proved analogously.

Suppose that, for any $j = 1, 2, \ldots$, there are $x_j \in D \cap \mathcal{N}$ and $r_j > 0$ with $B_{2r_j}(x_j) \subset \Omega$ such that

$$0 < v_j(x_j) \le \frac{r_j}{j}. \tag{7.9}$$

Now, define $\tilde{v}_j(x) := \frac{v_j(x_j + r_j x)}{r_j}$ and $\tilde{u}_j(x) = \frac{u(x_j + r_j x)}{r_j}$, for any $x \in B_1$. We recall that (3.17) implies that $\tilde{u}_j$ is a minimizer for J in B_1. So, it follows from P.1 in Proposition 3.4, Caccioppoli's inequality and Theorem A that

$$\int_{B_1} |\nabla \tilde{u}_j^{\pm}|^p \le C(N)\int_{B_2} (\tilde{u}_j^{\pm})^p \le C(N)\omega_N 2^{N+2p}L^p, \tag{7.10}$$

where L is the constant introduced in Theorem A.

Also, we observe that $\Delta_p \tilde{v}_j = 0$ in B_1 and that $\tilde{v}_j = \tilde{u}_j$ on ∂B_1. In particular, $\int_{B_1} |\nabla \tilde{v}_j|^p \le \int_{B_1} |\nabla \tilde{u}_j|^p$. This and (7.10) imply that $\|\tilde{v}_j\|_{W^{1,p}(B_1)} \le C(N)L^p$, up to renaming $C(N)$ (recall that $\tilde{u}_j^{\pm}$ are p-subharmonic, thanks to P.1 in Proposition 3.4).

Moreover, from the local regularity theory for $p-$harmonic functions we have that $\tilde{v}_j$ are uniformly $C^{1,\alpha}$ in $B_{\frac{1}{2}}$. Consequently, we have that there is a subsequence (still denoted by $\tilde{v}_j$) such that $\tilde{v}_j \to v_0$ weakly in $W^{1,p}(B_1)$ and uniformly in $B_{\frac{1}{2}}$, as $j \to +\infty$. In particular, by (7.9),

$$v_0(0) = \lim_{j\to\infty} \tilde{v}_j(0) = 0.$$

As for the sequence $\tilde{u}_j$, from (3.17) and Lemma 3.1 we infer that there is a subsequence (still denoted by $\tilde{u}_j$) such that $\nabla\tilde{u}_j \to \nabla u_0$ strongly in $L^q(B_1)$ for any $q>1$ and $\tilde{u}_j \to u_0$ uniformly in $\overline{B_1}$, as $j\to+\infty$. Furthermore, u_0 is a minimizer of J and from the convergence of traces it follows that $v_0 = u_0$ on ∂B_1. Also, by Corollary 3.6 we have that $u_0 \neq 0$, and by Proposition 3.4 we have that u_0 is $p-$subharmonic in B_1.

Altogether, we have obtained that

$$\Delta_p v_0 \leq \Delta_p u_0 \ \text{ in } B_1, \quad v_0 = u_0 \ \text{ on } \partial B_1 \quad \text{ and } v_0(0) = u_0(0) = 0.$$

But this is a contradiction to the comparison principle for p-harmonic functions.

The second inequality of (7.8) can be proven analogously.

Now we are ready to finish the proof of (7.6). From (7.7) and (7.8) we have

$$\begin{aligned} \int_{B_r(x_0)} \Lambda\chi_{\{u\leq 0\}} &\geq \frac{\bar{c}\Theta}{r^p}\int_{B_r(x_0)} |u-v_r|^p \\ &\geq \frac{\bar{c}\Theta}{r^p}\int_{B_{\kappa r}(x_0)} |u-v_r|^p \end{aligned} \tag{7.11}$$

for $0<\kappa<1$ to be chosen later. Observe that by standard gradient estimates

$$|\nabla v_r(y)| \leq \frac{C}{1-\kappa}\frac{\sup_{B_r(x_0)}|v_r|}{r} \leq \frac{CL}{1-\kappa}, \quad y\in B_{\kappa r}(x_0),$$

up to renaming $C>0$, where the last inequality follows from the maximum principle and Theorem A. Therefore, for any $y\in B_{\kappa r}(x_0)$

$$\begin{aligned} |v_r(y)-u(y)| &\geq |v_r(0)-u(y)| - |v_r(y)-v_r(0)| \\ &\geq v_r(0) - |u(y)| - |v_r(y)-v_r(0)| \\ &\geq v_r(0) - 2L\kappa r - \frac{CL}{1-\kappa}\kappa r \\ &\geq r\left(\Theta - \kappa L\left(2+\frac{C}{1-\kappa}\right)\right) \\ &\geq r\frac{\Theta}{2} \end{aligned} \tag{7.12}$$

if we choose κ small enough. Returning to (7.11) we finally get that

$$\frac{|\{u\leq 0\}\cap B_r(x_0)|}{|B_r(x_0)|} \geq \frac{\bar{c}\Theta^{p+1}}{2^p\Lambda}\kappa^N.$$

This finishes the proof of Lemma 7.3. □

8. Blow-up sequence of u, end of proof of Theorem B

In this section we study the blow-up sequences of a minimizer of (1.1) and prove a simple compactness result, that we use to conclude the proof of Theorem B. For this, let u be a minimizer of J and $x_0\in\partial\{u>0\}$. Consider a sequence of balls $B_{\rho_k}(x_0)$, with $\rho_k\to 0$. We call the sequence of functions defined by

$$u_k(x) = \frac{u(x_0+\rho_k x)}{\rho_k} \tag{8.1}$$

the blow-up sequence of u with respect to $B_{\rho_k}(x_0)$. Clearly u_k is also a local minimizer.

Proposition 8.1. *Let $x_0 \in \mathcal{N}$ and u_k be a blow-up sequence. Then there is a blow-up limit $u_0 : \mathbb{R}^N \to \mathbb{R}$ with linear growth such that for a subsequence*

- $u_k \to u_0$ *in* $C^\alpha_{loc}(\mathbb{R}^N)$ *for any* $\alpha \in (0,1)$,
- $\nabla u_k \to \nabla u_0$ *weakly in* $W^{1,q}$ *for any* $q > 1$,
- $\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ *locally in Hausdorff distance,*
- $\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}}$ *in* $L^1_{loc}(\mathbb{R}^N)$.

Proof. The first and second claims follow from Lemma 3.1 and a customary compactness argument to show that the blow-up limit u_0 exists.

We recall the definition of Hausdorff distance:

$$d_{\mathcal{H}}(F,G) := \inf\left\{\delta : F \subset \bigcup_{x\in G} B_\delta(x), G \subset \bigcup_{x\in F} B_\delta(x)\right\}.$$

Let $B_r := B_r(z_0)$ be a ball not intersecting $\partial\{u_0 > 0\}$. If $u_0 > 0$ in $\overline{B_r}$ then, by locally uniform convergence, $u_k > 0$ in $B_{\frac{r}{2}}$, thus implying that $\partial\{u_k > 0\} \cap B_{r/2} = \emptyset$. As for the case $u_0 \leq 0$ in B_r, it follows from Proposition 3.5 that $\frac{1}{r} \fint_{B_r}(u_0^+) < \varepsilon$, for any small $\varepsilon > 0$. Thus, by the uniform convergence, we have that $\frac{1}{r}\fint_{B_r}(u_k^+) < \varepsilon$ if k is sufficiently large. From Proposition 3.5 we conclude that $u_k \leq 0$ in $B_{r/2}$. In both cases we infer that $\partial\{u_k > 0\}$ does not intersect $B_{r/2}$ if k is large enough.

Conversely, if B_r does not intersect $\partial\{u_k > 0\}$ for any large k, then either $u_k > 0$ in B_r or $u_k \leq 0$ in B_r. In the first case, u_k is $p-$harmonic in B_r and hence so is u_0. Consequently, either $u_0 > 0$ in B_r or $u_0 \equiv 0$ in B_r. Thus B_r does not intersect $\partial\{u_0 > 0\}$. In the second case, we have that $u_0 \leq 0$, so that again $\partial\{u_0 > 0\}$ does not intersect B_r.

Reasoning as above and using a covering argument one can show that, for a fixed compact set D, the quantity δ in the definition of $d_{\mathcal{H}}$, with $G = \partial\{u_0 > 0\} \cap D$ and $F = \partial\{u_k > 0\} \cap D$, can be chosen as small as we wish.

The last statement follows from the non-degeneracy of u^+ given by Corollary 3.6, the convergence of $\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ in Hausdorff distance and the fact that the N-dimensional Hausdorff measure $\mathcal{H}^N(\partial\{u_0 > 0\}) = 0$, since u_0 is also minimizer and Corollary 7.2 applies. Hence the proof of Proposition 8.1 is concluded. □

Remark 8.2. *In view of Proposition 3.5 we see that when we consider the blow-up of a minimizer, the limit cannot vanish, no matter how many times we blow-up the minimizer u at a non-flat point.*

We now finish the proof of Theorem B. More precisely, we show that

$$\mathcal{H}^{N-1}\left((\partial\{u > 0\} \setminus \partial_{\mathrm{red}}\{u > 0\}) \cap B_R\right) = 0. \tag{8.2}$$

First observe that $\mathcal{N} \subset \partial\{u > 0\} \setminus \partial_{\mathrm{red}}\{u > 0\}$, see the discussion in Section 5. Since the current boundary $T := \partial(\mathbb{R}^N \llcorner \{u > 0\} \cap B_R(0))$ is representable by integration, $\|T\| = \int_{B_R(0)} |D\chi_{\{u>0\}}|$, we get from Section 4.5.6. on page 478 of [17] that

$$\begin{aligned} &\partial\{u > 0\} \setminus \partial_{\mathrm{red}}\{u > 0\} = K_0 \cup K_+, \text{ where } \mathcal{H}^{N-1}(K_+) = 0 \\ &\text{and for } x_1 \in K_0,\ r^{1-N}\mathcal{H}^{N-1}(\partial_{\mathrm{red}}\{u > 0\} \cap B_r(x_1)) \to 0 \text{ as } r \to 0. \end{aligned} \tag{8.3}$$

Let us show that

$$K_0 = \emptyset. \tag{8.4}$$

To see this, for $k \in \mathbb{N}$, we define $u_k(x) := \frac{u(x_1+r_k x)}{r_k}$, where $r_k \to 0$ as $k \to +\infty$. By the compactness properties obtained in Proposition 8.1, we have that $u_k \to u_0$, as $k \to +\infty$, for some function u_0 and, for any test function φ,

$$\begin{aligned}\int_{B_R} \chi_{\{u_0>0\}} \operatorname{div} \varphi \longleftarrow \int_{B_R} \chi_{\{u_k>0\}} \operatorname{div} \varphi = r_k^{1-N} \int_{B_{Rr_k}(x_1)} \chi_{\{u>0\}} \operatorname{div} \varphi \left(\frac{x-x_1}{r_k}\right) \\ \leq \sup \varphi \, r_k^{1-N} \mathcal{H}^{N-1}(\partial_{\rm red}\{u>0\} \cap B_{Rr_k}(x_1)) \longrightarrow 0 \qquad \text{as} \qquad k \to +\infty,\end{aligned}$$

where (8.3) was also used.

Hence we infer that $\chi_{\{u_0>0\}}$ is a function of bounded variation which is constant a.e. in B_R. The positive Lebesgue density property of $\{u \leq 0\}$ obtained in Lemma 7.3 and translated to u_0 through compactness, and the strong maximum principle for $p-$harmonic functions demand u_0 to be zero. This is in contradiction with the non-degeneracy of u^+ stated by Proposition 3.5 (notice that, by a compactness argument, the non-degeneracy property translates to u_0). Thus (8.4) is proved.

From (8.3) and (8.4) we obtain that $\mathcal{H}^{N-1}\left((\partial\{u>0\} \setminus \partial_{\rm red}\{u>0\}) \cap B_R\right) = 0$. The proof of Theorem B is then finished.

9. Proof of Theorem C

With the aid of Theorem A, in this section we complete the proof of Theorem C.

Proof of Theorem C. It is well-know that in order to prove the estimate (2.5) it is enough to show that u grows linearly away from the free boundary. For this, let $0 \in \partial\{u>0\}$ and $B_{\frac14} \Subset \Omega$. Notice that, if for all $k \in \mathbb{N}$, $k \geq 2$, we have that $h(0, 2^{-i}) \geq h_0 2^{-i-1}$, then it follows from Theorem A that $\sup_{B_r} |u| \leq 2Lr$. Therefore, suppose that there is $k_0 \in \mathbb{N}$ such that

$$h\left(0, \frac{1}{2^j}\right) \geq \frac{h_0}{2} \frac{1}{2^j}, \quad j = 2, \dots, k_0 - 1, \tag{9.1}$$

but

$$h\left(0, \frac{1}{2^{k_0}}\right) < \frac{h_0}{2} \frac{1}{2^{k_0}}. \tag{9.2}$$

From (9.1) and Proposition 6.1 (or Corollary 6.2) it follows that

$$\sup_{B_{\frac{1}{2^{k_0-1}}}} |u| = \sup_{B_{\frac12 \frac{1}{2^{k_0-2}}}} |u| \leq 2L \frac{1}{2^{k_0-2}} = \frac{4L}{2^{k_0-1}}. \tag{9.3}$$

Denote $R_0 := \frac{1}{2^{k_0-1}}$ and introduce

$$v_0(x) := \frac{u(R_0 x)}{R_0}, \quad x \in B_1, \tag{9.4}$$

then by (3.17) it follows that v_0 is a minimizer in B_1. Furthermore, (9.3) yields

$$\sup_{B_1} |v_0| \leq 4L \tag{9.5}$$

and by (9.2) we see that $\partial\{v_0>0\} \cap B_{\frac12}$ is $h_0/2$ flat. Therefore, we infer from the second part of Theorem A that there are $\delta \in (0, \frac12)$ and $\alpha > 0$ depending on N, p, Λ, h_0 and $4L$ such

that $\partial\{v_0 > 0\} \cap B_\delta$ is $C^{1,\alpha}$ regular. Applying the boundary gradient estimates for $p-$harmonic functions we finally obtain

$$\sup_{B_{\frac{\delta}{2}}} |\nabla v_0^{\pm}(x)| \leq 4LC_0 \tag{9.6}$$

for some tame constant $C_0 > 0$. Recalling (9.4) and (9.3) we conclude that

$$\sup_{B_r} |u| \leq \frac{16L}{\delta} r, \quad \forall r < r_0$$

for some small universal constant r_0. This completes the proof of Theorem C. □

Appendix A. Viscosity solutions and linear development

Here we show Lemma 4.3.

Proof of Lemma 4.3. We first show a). Without loss of generality, we may assume that $x_0 = 0$ and $\nu = e_N$. Let $B := B_R(y_0)$ be a touching ball at $0 \in \Gamma$, for some $y_0 \in \Omega$ and $R > 0$.

Now, we want to establish (4.4). For this, we first construct a function that can be used as a barrier to control u from below in the ring $B_R(y_0) \setminus B_{R/2}(y_0)$. We consider the scaled $p-$capacitary function H, that is p-harmonic in $B_R(y_0) \setminus B_{R/2}(y_0)$, that vanishes on $\partial B_R(y_0)$ and that is equal to 1 on $\partial B_{R/2}(y_0)$. Observe that near the origin

$$H(x) = c(N,R)\, x_N + o(|x|), \tag{A.1}$$

for some $c(N,R) > 0$.

Using the Harnack inequality we see that $u(x) \geq c_0 u(y_0)$ in $\bar{B}_{R/2}(y_0)$, for some $c_0 > 0$. Thus, multiplying H with a suitable constant $\sigma := c_0 u(y_0)$ we obtain that $\sigma H \leq u$ on $\partial B_{R/2}(y_0)$. Moreover, $u \geq 0 = \sigma H$ on $\partial B_R(y_0)$. Hence, by comparison principle, we get that

$$u(x) \geq \sigma H(x) \quad \text{ in } B_R(y_0) \setminus B_{R/2}(y_0).$$

From this and (A.1) we obtain that

$$u(x) \geq \sigma c(N,R) x_N + o(|x|) \tag{A.2}$$

near the origin.

Now we take k_0 the smallest positive integer such that $2^{-k_0} \leq R/2$ and we define

$$A_k := \sup\{m : u(x) \geq m x_N \text{ in } B_{2^{-(k_0+k)}} \cap B_R(y_0)\}, \qquad k = 0, 1, 2, \ldots. \tag{A.3}$$

Thanks to (A.2) the set of numbers m in the definition of A_k is not empty. Notice also that the sequence $\{A_k\}$ is increasing, and so we let $A := \sup A_k$.

We observe that

$$A > 0. \tag{A.4}$$

Indeed, since $u(x) \geq \sigma H(x)$ in $\overline{B_R(y_0) \setminus B_{R/2}(y_0)}$ then $A_0 > 0$. This implies (A.4), because A_k is increasing.

If $A = \infty$, then u grows faster than any linear function at 0. While, if $A < \infty$, then (4.4) holds true.

Now we claim that equality in (4.4) holds in any non-tangential domain. In what follows we denote by

$$\mathcal{B} := B_s(e_N/2), \quad \text{for some small } s>0, \quad \mathcal{D}_k := B_{R/r_k}(y_0/r_k)\cap B_1 \quad \text{and} \quad r_k := 2^{-(k+k_0)}. \tag{A.5}$$

If the claim fails then there exist a sequence of points $x^k \in B_R(y_0)$ and $\delta_0>0$ such that

$$u(x^k) > Ax^k_N + \delta_0|x^k| \qquad \text{and} \qquad |x_k| = r_k \sim \operatorname{dist}(x^k, \partial B_R(y_0)). \tag{A.6}$$

Now let $u_k(x) := \frac{u(r_k x)}{r_k}$. Notice that (A.6) implies that

$$u_k(y^k) > Ay^k_N + \delta_0,$$

where $y^k := x^k/r_k \in \partial B_1 \cap B_{R/r_k}(y_0/r_k)$. This implies that

$$u_k(x) - Ax_N \ge c_0\,\delta_0 \tag{A.7}$$

on some fixed portion of $\partial B_1 \cap B_{R/r_k}(y_0/r_k)$, for some $c_0>0$. So, (A.7) and the Harnack inequality give that

$$u_k(x) - Ax_N \ge \frac{c_0\delta_0}{100} \quad \text{in } \mathcal{B}, \tag{A.8}$$

where $\mathcal{B}$ has been introduced in (A.5) and we can take $s=\frac{1}{8}$.

Since u_k are uniformly $C^{1,\alpha}_{loc}$ in $B_2\cap B_{R/r_k}(y_0/r_k)$ and uniformly continuous in $B_2\cap\{x_N>0\}$, we have that, up to a subsequence, u_k converges uniformly to some $u_0\ge 0$ in $B_1\cap\{x_N\ge 0\}$. Therefore, by construction of A,

$$u_0(x) - Ax_N \ge 0 \quad \text{in } \partial B_1\cap\{x_N\ge 0\}. \tag{A.9}$$

We recall (A.5) and define functions w_k as solutions to the following boundary value problem

$$\begin{cases} \Delta_p w_k = 0 & \text{in } \mathcal{D}_k\setminus\mathcal{B},\\ w_k = A_k x_N + \frac{c_0\delta_0}{200} & \text{on } \partial\mathcal{B},\\ w_k = A_k x_N & \text{on } \partial\mathcal{D}_k. \end{cases} \tag{A.10}$$

Now from the definition of A_k in (A.3) we have that $u_k(x)\ge A_k x_N$ in $B_{R/r_k}(y_0/r_k)\cap B_1 = \mathcal{D}_k$, and so $w_k = A_k x_N \le u_k$ on $\partial\mathcal{D}_k$. Moreover, on $\partial\mathcal{B}$ we have that $w_k = A_k x_N + \frac{c_0\delta_0}{200} \le Ax_N + \frac{c_0\delta_0}{200} \le u_k$, thanks to (A.8). By comparison principle we get that $w_k\le u_k$ in $\mathcal{D}_k\setminus\mathcal{B}$.

By construction $w_k\to w_0$ uniformly in $B_\mu\cap\mathcal{D}_k$ and

$$\|w_k - w_0\|_{L^\infty(\mathcal{D}_k)} \le \varepsilon_k \to 0. \tag{A.11}$$

From the stability of $C^{1,\alpha}$ norm in $B^+_{3/16}$ (recall that we chose $s=1/8$ in (A.5)) we conclude

$$\|\nabla(w_k - w_0)\|_{C^\alpha(B^+_{3/16}\cap\mathcal{D}_k)} \le \varepsilon_k. \tag{A.12}$$

This allows to estimate the Hölder norm of $\nabla(w_k-w_0)$ near the flat portion of the boundary of $B^+_{3/16}$.

By Hopf's Lemma there is $\gamma > 0$ such that $w_0 \geq (A+\gamma)x_N + o(|x|)$ near the origin. Combining we get that

$$\begin{aligned} u_k &\geq w_k = w_k - w_0 + w_0 \geq w_k - w_0 + (A+\gamma)x_N + o(|x|) \\ &\geq (A+\gamma)x_2 + o(|x|) - \varepsilon_k x_N \\ &\geq (A+\frac{\gamma}{2})x_N + o(|x|) \\ &\geq (A+\frac{\gamma}{4})x_N \end{aligned} \tag{A.13}$$

in $B^+_{1/2} \cap B_{R/r_k}(y_0/r_0)$. Returning to u we get that

$$u(x) \geq (A+\frac{\gamma}{4})x_N \geq (A_{k+1} + \frac{\gamma}{8})x_N$$

in $B_{r_k/2} \cap B_R(y_0) = B_{r_{k+1}} \cap B_R(y_0)$. This is a contradiction with the definition of A_k in (A.3).

Hence, (4.4) holds true in any non-tangential domain, and this concludes the proof of part a).

Now we show part b). For this, we take a ball $B_R(y_0)$ touching x_0 from outside Ω. We construct the barrier as follows: we let η to be a p-harmonic function in $B_{2R}(y_0) \setminus B_R(y_0)$, such that $\eta = 0$ on $\partial B_R(y_0)$ and $\eta = \max_{\partial B_{2R}(y_0)} u$ on $\partial B_{2R}(y_0)$. Then, from comparison principle we have that $u \leq \eta$ in $B_{2R}(y_0) \cap \Omega$. Moreover, by Hopf's Lemma

$$\eta(x) = C(N,R)x_N + o(|x|) \tag{A.14}$$

near the origin, for some $C(N,R) > 0$.

We take k_0 to be the smallest positive integer such that $2^{-k_0} < R/2$, and we define

$$\beta_0 := \inf\{m : m\eta(x) \geq u(x) \text{ in } B_{2^{-k_0}} \cap B^c_R(y_0)\},$$

and, for any $k \geq 1$,

$$\beta_k := \inf\{m : m\eta(x) \geq u(x) \text{ in } B_{2^{-(k_0+k)}} \cap B^c_R(y_0)\}.$$

Since β_k is a decreasing sequence, we can take $\tilde{\beta} := \inf \beta_k$. Hence, $\tilde{\beta} \geq 0$, and, setting $\beta := \tilde{\beta}C(N,R)$, from (A.14) we deduce (4.5).

In order to prove equality in (4.5) in every non-tangential domain, one can proceed as in the proof of part a). This concludes the proof of Lemma 4.3. □

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